
Introduction

Throughout our careers, certain contemporary mathematical problems have caught our interest because their origins lie in mathematics covered in the elementary school curriculum and their development can be traced through high school, college, and university level mathematics. This book is intended to provide a source for the mathematics (from beginning to advanced) needed to understand the emergence and evolution of five of these problems.

For example, one of the problems concerns the number of lattice points in regions in n -dimensional Euclidean space. The study of lattice point geometry begins in the early grades when students are introduced to lattice polygons and their properties by means of geoboards. This hands-on experience prepares the way for an expanded discussion of area in the upper elementary grades. Pick's theorem in the plane and lattice polyhedra in three dimensions are part of the high school curriculum. At the university level, these topics are extended to higher dimensions where theorems in number theory, such as the two squares theorem and the four squares theorem, play a role. Finally, at the upper undergraduate level and in graduate school, the study of results, such as Minkowski's theorem on lattice points in convex regions and Ehrhart's theorem (a higher-dimensional version of Pick's theorem), leads students directly into current research in mathematics.

The four other problems involve

1. the finiteness of the length of an elementary arithmetic game;
2. the search for a simple test to determine which positive integers are the areas of right triangles whose sides have rational length;

3. the determination of properties of real numbers from their rational approximations; and

4. the exploration of regions that can be assembled from all of the pieces obtained by cutting up another region.

The five problems are treated in five separate chapters. Each chapter begins with the elementary mathematics involved at its source and proceeds through development of these concepts to a discussion (with the exception of the Four Numbers problem) of important results in contemporary research. Two of the chapters begin with basic number theory (The Four Numbers Problem and Rational Approximation), two begin with elementary geometry (Lattice Point Geometry and Dissection), and one begins with a combination of elementary geometry and number theory (Rational Right Triangles). The introduction to each chapter summarizes the contents of its various sections as well as the background required.

The Four Numbers Problem which begins this book does not have the depth or substance of the other problems, but its analysis is rich in applications of linear algebra, number theory and probability distributions. It is an excellent example of the vertical development of a mathematical problem except that it stops short of current mathematics research.

Our book is designed for students and teachers of mathematics from high school through graduate school. It should also prove of interest to working mathematicians who are curious about mathematical results in fields other than their own. It can be used by teachers at all of the above mentioned levels for the enhancement of standard curriculum materials or extra-curricular projects. Serious students of mathematics can begin reading the book in high school or earlier, and penetrate more deeply into the advanced aspects of the problems as they gain more mathematical experience. Three of the problems discussed here are addressed at a lower level in our book *Trimathlon*, published by A. K. Peters, Ltd.

We have presented this material many times in talks to audiences at all of the levels mentioned above. It has always proved popular.

Acknowledgements

We acknowledge with gratitude the contributions to our book made by our colleagues, students, and friends over the span of twenty years or more. It is not possible to mention by name all those with whom we have had conversations about this material, but we express our appreciation to them here.

We thank Branko Grünbaum, Matam P. Murthy, Raghavan Narasimhan, and Madhav Nori for their advice and assistance.

Among the many students and former students who engaged in spirited mathematical discussions with us on these topics are Steve Balady, Robert Beals, Walter Carlip, Logan Chariker, David Cohen, Nick Ford, Gabe Gaster, Matt Gealy, Dan Gulotta, Joe Hundley, Chris Jeris, Greg Kupferberg, Nick Longo, Jamie Pommersheim, Keith Ramsay, Ralph Thomas, Jeremy Sugarman, Alex Volfovsky, and John Zekos.

We thank Sam Altschul, David Coley, Jonathan Nieder, and Sam Raskin for their careful, critical reading of portions of the manuscript.

We are grateful to John Boller, Sam Isaacson, Russell Miller, Nick Ramsey, Ryan Reich, Wei Ren, Travis Schedler, and Loren Spice for their incisive ideas on these problems, as well as their perceptive reading of portions of them.

We are in great debt to Mitya Boyarchenko, Jack Huizenga, and Calvin Lin who contributed significant mathematical ideas to our book as well as excellent advice to us.

Finally, we thank John Ewing and the American Mathematical Society for the warm reception of our manuscript, and Sergei Gelfand for his interest in it for many years.