Chapter 1

The Decoupling Principle

Many phenomena in mathematics, physics, and engineering are described by linear evolution equations. Light and sound are linear waves. Population growth and radioactive decay are described well by first order linear differential equations. To understand the behavior of nonlinear systems near equilibrium, one always approximates them with linear systems, which can then be solved exactly. In short, to understand nature you have to understand linear equations.

In this book we cover a variety of linear evolution equations, beginning with the simplest equations in one variable, moving on to coupled equations in several variables, and culminating in problems such as wave propagation that involve an infinite number of degrees of freedom. Along the way we develop techniques, such as Fourier analysis, that allow us to decouple the equations into a set of scalar equations that we already know how to solve.

The general strategy is always the same. When faced with coupled equations involving variables $x_1, \ldots, x_m$, we define new variables $y_1, \ldots, y_m$. These variables can always be chosen so that the evolution of y_1 depends only on y_1 (and not on $y_2, \ldots, y_m$), the evolution of y_2 depends only on y_2, and so on. See Figure 1.1. To find $x_1(t), \ldots, x_m(t)$ in terms of the initial conditions $x_1(0), \ldots, x_m(0)$, we convert $\mathbf{x}(0)$ to $\mathbf{y}(0)$, then solve for $\mathbf{y}(t)$, then convert to $\mathbf{x}(t)$.

The first and third (vertical) steps are pure linear algebra and require no knowledge of evolution. The second step is done one variable at a time, and requires us to understand scalar evolution equations, but does not involve

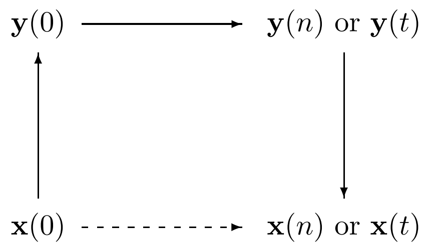

Figure 1.1. The decoupling strategy

linear algebra. Instead of dealing with coupled equations all at once, we consider the coupling and the underlying scalar equations separately.

Scalar linear evolution equations. The simplest linear evolution equations involve a single variable, which we will call x. Here are some examples of scalar linear evolution equations, which may already be familiar:

(1) **Discrete time growth or decay**. Suppose $x(n)$ is the number of deer in a forest in year n, and that $x(n)$ is proportional to $x(n-1)$. That is,

$$x(n) = ax(n-1), \tag{1.1}$$

for some constant a. Solving this equation we find that

$$x(n) = a^n x(0). \tag{1.2}$$

Of course, equation (1.1) could describe a variety of phenomena. The variable x could denote the amount of money in the bank, or the size of a radioactive sample, or indeed anything that has the unifying property: *How much you have tomorrow is proportional to how much you have today.* I leave it to you to think up additional examples.

(2) **Continuous time growth or decay**. If growth or decay is a continuous process (as is essentially the case with radioactive decay, or with population growth in humans, bacteria, and other species that never stop breeding), then equation (1.1) should be replaced by a linear differential equation:

$$\frac{dx}{dt} = ax. \tag{1.3}$$

The solution to this equation is

$$x(t) = ce^{at}, \tag{1.4}$$

where the constant c is determined by the initial condition:

$$c = x(0). \tag{1.5}$$

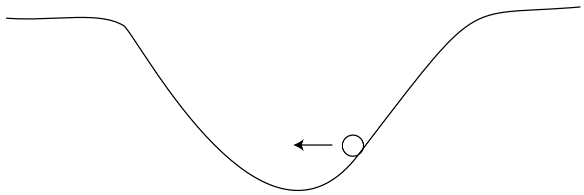

Figure 1.2. Acceleration is toward center

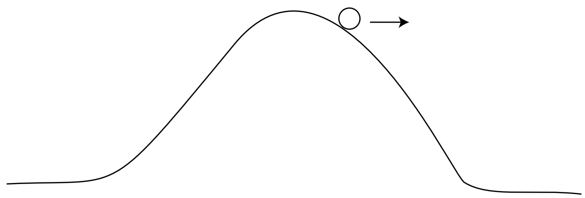

Figure 1.3. Acceleration is away from center

(3) **Oscillations near a stable equilibrium**. If x is the displacement of a mass on a spring, or the position of a ball rolling near the bottom of a hill (see Figure 1.2), or indeed *any* physical quantity near a stable equilibrium, then x is described by the second-order ordinary differential equation (ODE):

$$\frac{d^2}{dt^2} = -ax, \tag{1.6}$$

where a is a positive constant. This equation says that x is being accelerated back towards the origin at a rate proportional to x. Let $\omega = \sqrt{a}$. The general solution to equation (1.6) is

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t), \tag{1.7}$$

where c_1 and c_2 are determined by initial data. If our variable has initial value x_0 and initial velocity v_0, then

$$c_1 = x_0, \qquad c_2 = v_0/\omega. \tag{1.8}$$

(4) **Oscillations near an unstable equilibrium**. Let x be the position of a ball balanced near the *top* of a hill, as in Figure 1.3. The bigger x gets, the harder our ball gets pushed *away from* equilibrium. That is,

$$\frac{d^2x}{dt^2} = +ax, \tag{1.9}$$

for some positive constant a. This phenomenon goes by many names, from the technical "positive feedback loop" to the commonplace "vicious cycle". We all know of many real-world examples where a delicate balance can be ruined by a small push either way. Let $\kappa = \sqrt{a}$. The general solution to equation (1.9) is

$$x(t) = c_1 e^{\kappa t} + c_2 e^{-\kappa t}, \tag{1.10}$$

where once again c_1 and c_2 can be computed from the initial conditions.

Coupled linear evolution equations. Unfortunately, most real-world situations involve several quantities $x_1, x_2, \ldots, x_m$ coupled together. By "coupled" we mean that the evolution of each variable x_i depends on the other variables, not just on x_i itself.

For example, consider the population of owls and mice in a certain forest. Since owls eat mice, the more mice there are, the more the owls have to eat, and the more owls there will be next year. The more owls there are, the more mice get eaten, and the fewer mice there will be next year. If $x_1(n)$ and $x_2(n)$ describe the owl and mouse populations in year n, then the equations governing population growth might take the form

$$\begin{aligned} x_1(n) &= a_{11}x_1(n-1) + a_{12}x_2(n-1), \\ x_2(n) &= a_{21}x_1(n-1) + a_{22}x_2(n-1), \end{aligned} \tag{1.11}$$

where a_{11}, a_{12} and a_{22} are positive constants and a_{21} is a negative constant.

In such situations it is convenient to combine the variables x_1 and x_2 into a vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and combine the coefficients $\{a_{ij}\}$ into a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{1.12}$$

Equations (1.11) can then be rewritten as

$$\mathbf{x}(n) = A\mathbf{x}(n-1). \tag{1.13}$$

Just as equation (1.13) is a multivariable extension of equation (1.1), there are multivariable extensions of equations (1.3), (1.6) and (1.9), namely

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \tag{1.14}$$

$$\frac{d^2\mathbf{x}}{dt^2} = -A\mathbf{x}, \tag{1.15}$$

and

$$\frac{d^2\mathbf{x}}{dt^2} = +A\mathbf{x}, \tag{1.16}$$

respectively. Of course, in these equations $\mathbf{x}$ does not have to be a 2-component vector. $\mathbf{x}$ could be an m-component vector, in which case A would be an $m \times m$ matrix of coefficients. In fact, $\mathbf{x}$ could even be an element of an infinite dimensional vector space, with A an operator on that space.

Solving equations (1.13)–(1.16) certainly looks more difficult than solving the corresponding 1-variable equations, but there is a common occurrence where it is equally easy.

Decoupled equations. Suppose the matrix A is diagonal. For example, suppose that

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}. \tag{1.17}$$

Then equation (1.13) becomes

$$\begin{aligned} x_1(n) &= 2x_1(n-1), \\ x_2(n) &= 3x_2(n-1). \end{aligned} \tag{1.18}$$

Instead of having one matrix equation (or equivalently two coupled scalar equations), we have two *uncoupled* scalar equations, and we can immediately write down the solutions:

$$x_1(n) = 2^n x_1(0); \qquad x_2(n) = 3^n x_2(0). \tag{1.19}$$

It is similarly easy to solve equations (1.14)–(1.16). What an improvement from the coupled situation!

Decoupling coupled equations. Next, we consider an example in which two equations *look* coupled, but with a judicious change of coordinates, can be changed into two uncoupled equations. Consider the equations

$$\begin{aligned} x_1(n) &= 2x_1(n-1) + x_2(n-1), \\ x_2(n) &= x_1(n-1) + 2x_2(n-1). \end{aligned} \tag{1.20}$$

These equations are of the form (1.13) with

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{1.21}$$

We now define new variables

$$y_1 = (x_1 + x_2)/2; \qquad y_2 = (x_1 - x_2)/2, \tag{1.22}$$

so that

$$x_1 = y_1 + y_2; \qquad x_2 = y_1 - y_2. \tag{1.23}$$

Adding the two equations of (1.20) and dividing by 2 gives

$$y_1(n) = 3y_1(n-1), \tag{1.24}$$

while taking the difference of the two equations of (1.20) and dividing by 2 gives

$$y_2(n) = y_2(n-1). \tag{1.25}$$

The coupled equations $\mathbf{x}(n) = A\mathbf{x}(n-1)$ have been converted to decoupled equations

$$\mathbf{y}(n) = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{y}(n-1). \tag{1.26}$$

The decoupled equations can be solved by inspection, with solutions

$$y_1(n) = 3^n y_1(0); \qquad y_2(n) = y_2(0). \tag{1.27}$$

To solve the original equations (1.20), we follow the 3-step procedure of Figure 1.1. We first use (1.22) to convert the initial data $\mathbf{x}(0)$ into initial data $\mathbf{y}(0)$. We then use equations (1.27) to find $\mathbf{y}(n)$. Finally, we use equations (1.23) to compute $\mathbf{x}(n)$. The end result is

$$\begin{aligned} x_1(n) &= \frac{1}{2}(3^n+1)x_1(0) + \frac{1}{2}(3^n-1)x_2(0), \\ x_2(n) &= \frac{1}{2}(3^n-1)x_1(0) + \frac{1}{2}(3^n+1)x_2(0). \end{aligned} \tag{1.28}$$

Using the y coordinates, one can similarly solve equations (1.14)–(1.16).

The moral of the story. The combinations $y_1 = (x_1 + x_2)/2$ and $y_2 = (x_1 - x_2)/2$, in terms of which everything simplified, were not chosen at random. The vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are the *eigenvectors* of the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, while 3 and 1 are the corresponding *eigenvalues*. In general, whenever we can find m eigenvalues (and corresponding eigenvectors) of an $m \times m$ matrix A, we can convert the coupled system of equations (1.13) (or (1.14) or (1.15) or (1.16)) into m decoupled equations, which we can then solve, as above. (In Section 4.5 we determine when it is possible to find enough eigenvalues and eigenvectors.) The same ideas also apply (with a few modifications) to infinite dimensional problems, such as heat flow, wave propagation, and diffusion.

Note that the combinations $(x_1 + x_2)/2$ and $(x_1 - x_2)/2$ were the same for all four types of problems. In general, decoupling a system of equations does not require us to know anything about the class of equations; we just need to understand the eigenvalues and eigenvectors of the matrix A. This is the Decoupling Principle. Only after we have decoupled the variables do we need to examine the resulting (scalar!) equations.

This book is an extended exploration of this principle. After a review of elementary linear algebra in Chapters 2 and 3, we learn about eigenvalues and eigenvectors in Chapter 4. In Chapter 5 we use eigenvalues and eigenvectors to solve the kinds of problems we have just discussed. Thereafter we consider systems with additional structures, such as an inner product or a complex structure. We shall see that many powerful techniques, such as Fourier Analysis, are just special applications of the Decoupling Principle.

I have attempted to present each concept in three settings. The first setting is in $\mathbb{R}^n$, where the problem is essentially a matrix computation. The second setting is in a general n-dimensional vector space, where a choice of basis reduces the problem to one on $\mathbb{R}^n$ (or $\mathbb{C}^n$). The third setting is in an infinite dimensional vector space. The goal is not to understand the full theory of infinite dimensional vector spaces (something well beyond the scope of this book), but to see enough examples to start building up intuition. Infinite dimensional spaces appear with increasing frequency later in the book.

Exercises

(1) With $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, write down the solution to equation (1.14). That is, express $x_1(t)$ and $x_2(t)$ in terms of initial conditions $x_1(0)$ and $x_2(0)$.

(2) With $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, write down the solution to equation (1.15). That is, express $x_1(t)$ and $x_2(t)$ in terms of initial data (values and first derivatives at $t = 0$).

* (3) With $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, write down the most general solution to equation (1.16).

* (4) Let $x_1(0) = 5$ and let $x_2(0) = 3$. Solve equations (1.20) in the three steps indicated, first computing $y_1(0)$ and $y_2(0)$, then computing $y_1(n)$ and $y_2(n)$, and then computing $x_1(n)$ and $x_2(n)$. Check that your results agree with equation (1.28).

(5) With $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, solve equation (1.14). That is, express $x_1(t)$ and $x_2(t)$ in terms of initial conditions $x_1(0)$ and $x_2(0)$.

* (6) With $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, solve equation (1.15). That is, express $x_1(t)$ and $x_2(t)$ in terms of initial data (values and first derivatives at $t = 0$).

* (7) With $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, find the most general solution to equation (1.16). You need not express $\mathbf{x}(t)$ in terms of initial data.

Exploration: Beats

(1) Let ϵ be a small positive number (say, 0.1), and consider the coupled differential equations

$$\begin{aligned}\frac{d^2x_1}{dt^2} &= -x_1 + \epsilon x_2,\\ \frac{d^2x_2}{dt^2} &= \epsilon x_1 - x_2.\end{aligned} \tag{1.29}$$

MATLAB, Mathematica and Maple all have built-in algorithms for solving such differential equations numerically. Using technology, solve these equations with initial conditions $x_1(0) = 1$, $x_2(0) = 0$, $\dot{x}_1(0) = \dot{x}_2(0) = 0$, and plot both x_1 and x_2 as a function of time. Qualitatively, how does the system behave when $t \ll 1/\epsilon$? How does it behave when t gets larger than $1/\epsilon$? Make sure to plot your solutions at least to $t = 2\pi/\epsilon$.

(2) Next, vary the size of ϵ. What changes? What doesn't change?

(3) Now run the system with initial conditions $x_1(0) = x_2(0) = 1/2$, $\dot{x}_1(0) = \dot{x}_2(0) = 0$. Plot x_1, x_2, $(x_1 + x_2)/2$ and $(x_1 - x_2)/2$ as a function of time. What is going on? Try the same thing with the initial conditions $x_1(0) = 1/2$, $x_2(0) = -1/2$, $\dot{x}_1(0) = \dot{x}_2(0) = 0$.

(4) The vector $\begin{pmatrix}1\\0\end{pmatrix}$ can be viewed as the sum of $\begin{pmatrix}1/2\\1/2\end{pmatrix}$ and $\begin{pmatrix}1/2\\-1/2\end{pmatrix}$. How does the solution you found in step 1 compare with the sum of the two solutions you found in step 3?

(5) By doing a change of variables from x_1, x_2 to $y_1 = (x_1 + x_2)/2$ and $y_2 = (x_1 - x_2)/2$, you can convert equations (1.29) into decoupled equations. Solve these by hand to understand your results in steps 1–4.

If you get stuck on any of these steps, you can look ahead to Section 5.3, where a physical model involving these equations is analyzed in depth.