

Preface

Hypocrite lecteur,—mon semblable,—mon frère!

[Hypocrite reader,—my fellow creature,—my brother!]

Charles Baudelaire, 1861, in: *Les Fleurs du Mal* [Bau,p.16]

This book (or essay) is the result of more than fifteen years of reflexion and research on or around the subject mentioned in the primary title, *In Search of the Riemann Zeros*. We focus on the quest for the ultimate meaning and justification of the celebrated *Riemann Hypothesis*, perhaps the most vexing and daunting problem in the history of Mathematics.

As is well known, the Riemann Hypothesis (or Riemann's Conjecture) states that the complex zeros (also called the *Riemann zeros* in this book) of the Riemann zeta function $\zeta = \zeta(s)$ must all lie on the *critical line* $\text{Re } s = \frac{1}{2}$. This problem was furtively formulated in 1859 in Riemann's inaugural address to the Berlin Academy of Sciences. The latter is certainly one of Riemann's masterpieces as well as his only published paper dealing with number theory, specifically, the asymptotic properties of the prime numbers.

Riemann's Conjecture has so many desirable and important consequences in mathematics and beyond, and has become so engraved in our collective psyche, that few experts now doubt that it is true. Further, it has been numerically verified up to astronomical (albeit, finite) heights; i.e., for $|\text{Im } s| < T$, with T very large, no less than two trillion. In addition, counterparts of the Riemann Hypothesis in the simpler realm of finite geometries (technically, curves and higher-dimensional varieties over finite fields) have been firmly established about 50 and 30 years ago by André Weil and Pierre Deligne, respectively, thereby providing valuable insight into what might be true and which structures should be expected in the much more complex and elusive arithmetic realm of the original conjecture. In particular, the old Pólya–Hilbert dream of finding a suitable *spectral interpretation* for the Riemann zeros has found a natural place in this setting. Whether or not it can be turned into a successful proof of the Riemann Hypothesis still remains to be seen.

More recently, further evidence towards such a spectral interpretation has been discovered in a different and seemingly unrelated context. It relies on intriguing and still quite mysterious analogies between the statistics of atomic or molecular (quantum mechanical) spectra and that of the average spacing between the Riemann zeros along the critical line. This is now part of random matrix theory, a fascinating subject which will not be much discussed here but about which the interested reader will be able to find several references in the text.

Finally, and most importantly, as is often the case in mathematics, the simplicity and aesthetic quality of Riemann's Conjecture is perhaps the most powerful

argument in its favor. Indeed, as is well known and will be further explained in the introduction, the Riemann Hypothesis can be poetically (but rather accurately) reformulated as stating that \mathbb{Q} , the field of rational numbers, lies as harmoniously as possible within the field of real numbers, \mathbb{R} . Since the ring of integers, \mathbb{Z} —and hence, its field of fractions, \mathbb{Q} —is arguably the most basic and fundamental object of all of mathematics, because it is the natural receptacle for elementary arithmetic, one may easily understand the centrality of the Riemann Hypothesis in mathematics and surmise its possible relevance to other scientific disciplines, especially physics. (We note that for some physicists, only \mathbb{Q} truly exists. Yet, in practice as well as in theory, all measurable quantities are given by real numbers, not just by rational numbers.)

One of our original proposals in this book is that a helpful clue for unravelling the Riemann Hypothesis may come from surprising and yet to be fully unearthed or understood connections between different parts of contemporary mathematics and physics. This may eventually result in a unification of aspects of seemingly disparate areas of knowledge, from prime number theory to fractal geometry, noncommutative geometry, arithmetic geometry and string theory.

A *fil d'Ariane* (or connecting thread) throughout our present search has been provided by the striking analogies between the key symmetry of the Riemann zeta function (and its many number theoretic counterparts), as expressed analytically by a *functional equation*, and the various *dualities* exhibited by string theories in theoretical physics. (For simplicity and due to our own limitations, we will focus primarily in this book on only one such notion of duality, called *T-duality*.)

One of the author's long-term dreams would be to use such analogies to deduce something seemingly intractable—such as the conjectured location of the Riemann zeros on the critical line—from a much simpler fact on the other side of the mirror (say, from within the region $\text{Re } s > 1$, where both the series and the Euler product defining $\zeta(s)$ converge). Similarly, string theoretic dualities, in their multiple forms, are often used to transform an apparently impossible problem into one that is more transparent and much simpler to solve within the dual (or mirror) string theory.

Near the end of the main part of this book (Chapter 5), we will discuss a conjectural flow (called the *modular flow*) on the 'moduli space of fractal membranes'—along with its natural counterpart on the Riemann sphere, the *flow of zeros*—that would help realize this idea in a more abstract and global context.¹ In particular, conjecturally, it would enable us to explain why the Riemann Hypothesis is true. Moreover, it would show how seemingly very different fractal-like geometries and arithmetic geometries are all part of a common continuum, namely, the orbits of the modular flow. Accordingly, arithmetic geometries would represent the ultimate evolution of the modular flow (and also correspond to its stable and attractive fixed points). Similarly, the Riemann zeros would be the *attractor* of the flow of zeros (of zeta functions)—and hence, because of the aforementioned connections between symmetries and dualities, would have to lie on the critical line (or, equivalently, on the Equator of the Riemann sphere), as stated by the Riemann Hypothesis. Still conjecturally, an analogous reasoning would apply in order to understand and establish the Generalized Riemann Hypothesis, corresponding to other arithmetic geometries and to the critical zeros of their associated zeta functions.

¹As the subtitle of this book indicates, *Strings, fractal membranes and noncommutative spacetimes*, a substantial amount of preparation will be needed before we can reach that point.

We note that the cover of this book provides a symbolic depiction of the flows of zeta functions and of their zeros induced by the modular flow on the moduli space of fractal membranes. See also, respectively, Figures 1 and 2 near the beginning of Section 5.5.2.

As will be abundantly clear to the reader and is probably already apparent from the preceding discussion, this book is not a traditional mathematical research monograph.² In particular, we absolutely do not claim to provide a complete solution to the original enigma, let alone full proofs or even partial justifications for our main proposals and conjectures. At best, in many cases, we can only offer heuristic arguments based on mathematical or physical analogies. It should be plainly understood from the context (either in the text itself or in the notes) whether a given claim is a physical or heuristic statement, a reasonable expectation, a conjecture, a mathematical theorem, or neither. For example, at this stage, the existence of the modular flow and its expected properties are purely conjectural. They rely partly on analogies with physical theories and constructs (string theories and dualities, as reformulated in the language of vertex algebras and noncommutative geometry, conformal field theories, quantum statistical physics, renormalization group flow) and on mathematical concepts and theories (moduli spaces of quantized fractal strings, the author and his collaborators' theory of complex fractal dimensions, Deninger's spectral interpretation program and heuristic notion of 'arithmetic site', modular theory in operator algebras, and Connes' noncommutative geometry). On the other hand, as will be further discussed in the text (namely, in Section 4.2), the notion of a *fractal membrane* (or quantized fractal string) introduced in Chapter 3 of this book can now be put on a rigorous mathematical footing. As a result, other statements in Chapter 3 have become true theorems.

In some sense, this book should be viewed partly as a research program to pursue (rather than to complete) the above quest, and partly as a contribution to a continuing dialogue between mathematicians, physicists and other geometers of 'reality'. As such, it is written in multiple tongues, sometimes in mathematical language and sometimes in physical language. Appeals to both rigor and intuition alternate, in no particular order, without apparent rhyme or reason. Just as importantly, even within our more mathematical discussions, the boundaries between the traditional research areas are often blurred. This is one reason we have found it necessary to include a significant amount of background material, as evidenced by the large number of appendices in the second part of this book. If nothing else, and irrespective of our own specific goals, the reader may benefit from reading part of that material, which she can choose according to her own tastes and needs.

In advance, we ask the reader's indulgence and hope that she will approach this book with an open and flexible mind. Above all, we wish that, whether or not she agrees with the premises and primary message of the book, the reader will have an eventful and pleasurable journey, contemplating along the way glimpses of mathematical beauty and fruitfully interacting with its enduring reality.

Michel L. Lapidus

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²In fact, this is the primary reason why this author did not want it to be included in a regular book series.