Preface

It's been more than half a century since Alexander Grothendieck burst onto the mathematical scene. His natural gift for apt abstract generalisations was first tested in the arena of functional analysis and was not found wanting. His inborn compass led him to isolate notions that were to play a central role in the study and the development of Banach space theory to this very day.

He was the first to formulate isomorphic invariants of special Banach spaces by comparing these spaces with other Banach spaces via the bounded linear operators between them.

He caught and held the attention of those that could appreciate his ideas with concrete examples of enduring importance.

He recognized the importance of the nature and location of the finite dimensional subspaces of a space and utilized such — "local" theory was born.

He was the first analyst to seriously chase diagrams in the hopes of catching essential isomorphic characteristics of Banach spaces, and catch them he most certainly did.

Nowhere are these innovations more in evidence than in his infamous Résumé. Produced during his years in Sao Paulo, the Résumé sets forth Grothendieck's plan for the study of the finer structure of Banach spaces. He uses tensor products as a foundation upon which he builds the classes of operators most important to the study and establishes the importance of the "local" theory in the study of these operators and the spaces they act upon. When in the late sixties, Joram Lindenstrauss and Aleksander Pełczynski redressed his Fundamental Inequality in the trappings of operator ideals, it signaled the rebirth of Banach space theory. The ideas of the Résumé were demystified and made palatable to a generation of mathematical analysts. Banach space theory soon attracted a slew of talented young mathematicians who, with Lindenstrauss and Pełczynski at the lead, established the subject as a worthy aid in studying the problems of more classical aspects of mathematical endeavor, such as harmonic analysis, probability, complex analysis, geometry of convex bodies, real analysis and operator theory; at the same time the study of Banach space theory for its own sake became a worthwhile occupation.

To be sure, much of the success of the work of Lindenstrauss and Pełczynski is due to their shedding Grothendieck's Fundamental Inequality of its mystifying tensorial formulation. Nevertheless, they took note of what they had done. To wit, "Though the theory of tensor products constructed in Grothendieck's paper has its intrinsic beauty we feel that the results of Grothendieck and their corollaries can be more clearly presented without the use of tensor products. The paper of Grothendieck is quite hard to read and its results are not generally known even to experts in Banach space theory."

What they said in the late sixties still applies! However, we think the Résumé still has much to offer and we believe that its contents are still worthy of close study; it is to support such a view that we devote this work. Mainly we have followed the path blazed by Grothendieck; we have presented most of the arguments using the machinery available to him at the time of discovery. To be sure, Grothendieck knew what mattered to Banach space affairs in great detail and used his genius to put much of the best of that material to work efficiently and effectively in the execution of his plan. There are several junctures where "modern technology" might shorten some arguments; usually we relegate such to our Notes and Remarks.

To ensure a clear understanding of just what tools were available in Grothendieck's functional analysis days, we have included several appendices. Though we sometimes opt for a more modern presentation than was available in the fifties, we stay faithful to the formulation of the results as used by Grothendieck.

A brief outline of the contents follows.

In Chapter 1, we present in detail the basic facts and features of tensor norms, including how the integral bilinear forms and operators derive from a given tensor norm. This chapter has been presented to a number of defenseless graduate students throughout the years and the level of detail is a consequence. Recall the words of Professor C. A. Rogers, who in the Introduction to his treasured book on "Hausdorff Measures", acknowledges that his "book is largely based on lectures, and, as I like my students to follow my lectures, proofs are given in great detail; this may bore the mature mathematician, but I believe will be a great help to anyone trying to learn the subject ab initio." We have taken Professor Rogers' words as sound advice, particularly in light of the arid nature of the initial aspects of tensor norms.

Chapter 1, then, is devoted to the study of tensor norms and the operator ideals generated by them. We have inserted examples, as well as some simple computations, that we believe "will ease the pain" a bit. Again, with an eye to exposing the serious student to how the most classical tensor norms (the projective and injective norms) behave, we have ended Chapter 1 with an exposition of Grothendieck's treatment of the Dvoretsky-Rogers theorem. Here we see how in infinite dimensional Banach spaces that the collections of absolutely p-summable series and weakly p-summable series constitute vastly different collections when p is a real number larger than or equal to 1, we compare these collections to the projective and injective tensor products of the classical sequence spaces with a general Banach space of infinitely many dimensions.

In Chapter 2 the central role played by C-spaces and the L-spaces in Banach space theory is firmly established. First, we investigate integral operators, their remarkable characterization in terms of factorization, and the relationship between the differentiability of vector-valued measures and the nuclearity of certain operators acting on Lebesgue spaces. Along the way we find that integral operators into L-spaces are precisely those that take the closed unit ball of their domain into an order bounded set. We build on these peculiarities (and associated phenomena in C-spaces) to provide a platform for the discussion of injective and projective tensor norms; here the seemingly arid wasteland of Chapter 1 springs to life and we are rewarded for our careful work by the sharp characterizations of left, right, and two-sided injective/projective tensor norms in terms of integral forms and operators. We then apply this theory to look at how various injective and projective tensor norms are derived from a given tensor norm and take particular pleasure in pursuing the natural tensor norms. We close Chapter 2 with a partial table of the natural tensor norms, comparing those thus far encountered.

In Chapter 3, Hilbert spaces join in the fun. After establishing the existence of the Hilbertian tensor norm H and proving that the H-integral operators between two Banach spaces are precisely those that factor through a Hilbert space, H is shown to be injective and so is easily comparable to the injective hull of the projective tensor norm, the so-called pre-integral norm. We briefly investigate the hermitian H-forms and their compadres the hermitian H^*-forms; surprising measure-theoretic consequences are drawn. This is followed with the proof of what is now known as "the little Grothendieck theorem", which has as a corollary the fact that on the product of C-spaces the H-forms and H^*-forms coincide. We close the chapter relating the various classes of integral operators relative to the natural tensor norms with ideas from the classical theory of operators between Hilbert spaces. The Hilbert-Schmidt operators are shown to coincide with those operators between Hilbert spaces that factor through an L-space; alternatively they are shown to be precisely those that factor through a C-space. A remarkable consequence of the local character of the classes of integral operators appears herein: Every Hilbert space is simultaneously isomorphic to a subspace of an L-space and a quotient of a C-space. It is also shown that every operator from an L-space into a Hilbert space can be extended to any larger domain in a continuous linear fashion.

Chapter 4 is where the fundamental theorem of the metric theory of tensor products is first formulated. Following Grothendieck, we give a number of its consequences as well as his original proof.

Throughout the text we have included a number of Notes and Remarks which, we hope, round out the presentation. We have tried to stay to the point and that is, as we see it, to expose the Résumé. Following the main text, we have four appendices. The first discusses the solutions to the open problems listed by Grothendieck. Since the solutions to these problems involved notions that evolved later than the appearance of the Résumé, we have included a very brief Glossary of terms. With these terms in hand, the discussion of the problems ought to be sufficient to give an overview of the disposition of these problems and their solutions.

There follow three more appendices wherein we discuss results that Grothendieck used that are critical to the understanding of the Résumé but are somewhat scattered across the literature. Here we bow to the god of convenience, using modern expositions in order to ease the pain somewhat.

We would be remiss if we did not acknowledge the work of many others on tensor products and operator ideals that influenced our work. The alert reader will find commentary on the works of these mathematicians in our *Notes and Remarks*, which are scattered throughout these deliberations.

To be sure we make special mention here of the works of Amemiya and Shiga (1957), Lindenstrauss and Pełczyński (1968), Gilbert and Leih (1980), Pietsch (1980), and Defant and Floret (1993). Each has clarified for us many of the mysteries encountered in the *Résumé*.

A recent addition to the literature on tensor products is the charming book by RayRyan (2002) "Introduction to Tensor Products of Banach Spaces"; those who are neophytes in tensor products will find Ryan's treatment sympathetic and comforting.

The first author was the beneficiary of many informal tutorials from Dan Lewis throughout the seventies; during these sessions, Dan would construct diagrams from analytical data and show how to use them. Some (but not all) of these tutorials took. Regardless, thanks Dan.

We gained a great deal by detailed criticism of Chungsun Choi and Daniele Puglisi who read an earlier version of this manuscript. We owe to Ignacio Villaneuve, the Great Cartographer, directions to the elegant proof of Proposition 4.1.4 saving the persevering reader much pain (had they had to read our version of the same proof). The reviewers made comments that also improved the text in various places and Haskell Rosenthal gave us some invaluable criticism as well as letting us introduce herein his notion of C-LUST; he also provided us with Theorem A.2.4 to highlight the potential role to be played by C-LUST.

Through the years we have lectured on this subject matter at various institutions. We wish to thank the Mathematics Departments for their hospitality. We particularly wish to thank those Departments at the University of the Andes in beautiful Merida, Venezuela, Kent State University, North-West University (Potchefstroom) and the University of Pretoria.

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