Chapter 1

Introduction

We describe in detail the main topics discussed in this book, these being the theorems of Poncelet and Cayley, billiards in an ellipse, and double queues.

1.1. The Theorems of Poncelet and Cayley

We state and discuss the theorems of Poncelet and Cayley.

Poncelet's Theorem. *Let C and D be two ellipses, with D inside C. Suppose there is an n-sided polygon inscribed in C and circumscribed about D. Then for any other point of C, there exists an n-sided polygon, inscribed in C and circumscribed about D, which has this point for one of its vertices. (We occasionally refer to such polygons as circumscribed between C and D.)*

Figure 1.1 illustrates the case $n = 3$. The solid line triangle $P_0P_1P_2$ is assumed to be inscribed in C and circumscribed about D. Poncelet's theorem asserts that we then obtain another such triangle, starting from any point Q_0 in C.

We remark that when C and D are two concentric circles, with D inside C, Poncelet's theorem is obvious, as the circuminscribed polygon can be rotated within the circular ring bounded by C and D. The astonishing fact is that the theorem continues to hold when C

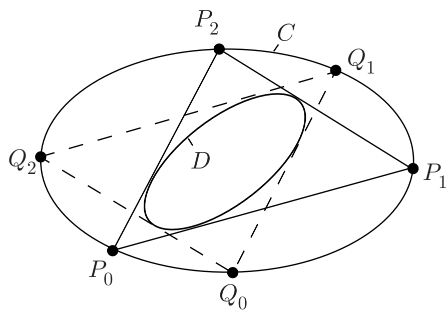

Figure 1.1

and D are not concentric circles or, more generally, two ellipses with C inside D.

There are several proofs of this remarkable theorem, none of which is elementary. Poncelet's theorem dates to the nineteenth century and has attracted the attention of many mathematicians of that period (a detailed historical account is given in [BKOR]). The main reason for this interest seems to stem from the fact that several proofs of this theorem require the use of complex and homogeneous coordinates, notions which were beginning to emerge at the time (1813) when Poncelet discovered his theorem.

Poncelet discovered the theorem while in captivity as war prisoner in the Russian city of Saratov. After his return to France, a proof appears in his book [P], published in 1822. The proof, which is synthetic and somewhat elaborate, reduces the theorem to two (not necessarily concentric) circles. A discussion of the ideas in Poncelet's proof is given in [BKOR], pp. 298–311.

Following Poncelet's proof, Jacobi gave an analytic proof based on identities from elliptic functions. Cayley's proof of his theorem is also based on such identities. In recent times, the two theorems have been studied anew by Griffiths and Harris (1977–1978), who gave modern algebro-geometric proofs.

In this book we adopt the approach of Griffiths and Harris [GH1, GH2], which relates the two theorems to elliptic curves (over the complex field) and to elliptic functions which form the function fields (set of meromorphic functions) of these curves. Thus, aside from their intrinsic interest, the theorems have the appeal of relating to a large body of mathematics. The required mathematics is covered in Part I on projective geometry—in particular, conics and their intersection theory; Part II on complex analysis—in particular, Riemann surfaces and elliptic functions; and Chapter 12 on dynamical systems.

As mentioned, the proofs of the Poncelet and Cayley theorems given in [GH1, GH2] use complex analysis. Consequently, we require a version of Poncelet's theorem in the complex projective plane P_2. This plane is the set of complex triples $x = (x_1, x_2, x_3)$, not all $x_i = 0$, any two triples considered to be equivalent if they have proportional coordinates. The ellipses C, D are now replaced by conics in P_2. This means that C, D have equations $C(x) = 0$, $D(x) = 0$, where $C(x)$ and $D(x)$ are quadratic forms in x with complex coefficients.

P_2-Version of Poncelet's Theorem. *Let C, D be smooth conics in general position. Suppose there is an n-sided polygon inscribed in C and circumscribed about D. Then for any point of C, there exists an n-sided polygon, also inscribed in C and circumscribed about D, which has this point for one of its vertices.*

The above statement requires elucidation, since it deals with conics in the complex plane. A conic with equation $C(x) = 0$ is called smooth if it has a tangent at each of its points (tangent lines are defined in §2.5). The smoothness condition is equivalent to $\det C \neq 0$, where $\det C$ is the determinant of the symmetric matrix associated with $C(x)$ (see §3.1). An n-sided polygon with vertices $p_1, \ldots, p_n$ is inscribed in C and circumscribed about D if the points $p_1, \ldots, p_n$ lie in C and the lines determined by the pairs of points (p_1, p_2), $(p_2, p_3), \ldots, (p_n, p_1)$ are tangent to D.

The conics C and D are in general position if they intersect in four points, which is the maximum number possible, by Bezout's theorem (see §4.1). Observe that in the original statement of Poncelet's theorem, the two ellipses do not intersect at all, thus seemingly contradicting the hypothesis of the P_2-version of Poncelet's theorem. The answer to this apparent contradiction is that the ellipses do intersect, usually in four points, but these points lie in P_2 and not in the real plane. An example illustrating this phenomenon is worked out in detail in §2.2.

The cases where C and D are not in general position are referred to as non-generic. They are described in Chapter 4. Poncelet's theorem also holds for the non-generic cases, but the proof requires some other considerations and is therefore presented in a separate chapter (Chapter 11).

In view of Poncelet's theorem, there arises the question of deciding from the equations for C and D whether there exists an n-sided polygon inscribed in C and circumscribed about D. Such a criterion is provided by Cayley's theorem, when the conics are in general position.

Let t be any complex number, and let $\det(tC+D)$ be the determinant of the quadratic form $tC(x)+D(x)$. For $t=0$, $\det(tC+D) = \det D \neq 0$. Hence $\sqrt{\det(tC+D)}$ has a power series in t, valid for small $|t|$.

Cayley's Theorem. *Let*

$$\sqrt{\det(tC+D)} = A_0 + A_1 t + \cdots + A_n t^n + \cdots .$$

There exists an n-sided polygon, inscribed in C and circumscribed about D, iff

$$\begin{vmatrix} A_2 & \cdots & A_{m+1} \\ \cdots & \cdots & \cdots \\ A_{m+1} & \cdots & A_{2m} \end{vmatrix} = 0, \text{ if } n = 2m+1 \ (m \geq 1);$$

$$\begin{vmatrix} A_3 & \cdots & A_{m+1} \\ \cdots & \cdots & \cdots \\ A_{m+1} & \cdots & A_{2m-1} \end{vmatrix} = 0, \text{ if } n = 2m \ (m \geq 2).$$

The theorems of Poncelet and Cayley are proved in Chapters 9 and 10, respectively. The proofs use a reformulation of Poncelet's theorem stated below. Let D^* be the dual of D, i.e., the set of lines tangent to D, and let

$$\mathcal{M} = \{(x,\xi) \colon x \in C, \xi \in D^*, x \in \xi\}.$$

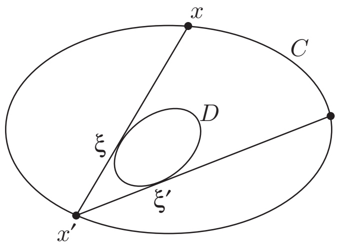

Figure 1.2

Let σ and τ be the two involutions (transformations of period two) of $\mathcal{M}$ defined by

$$\sigma(x, \xi) = (x', \xi),$$
$$\tau(x', \xi) = (x', \xi'),$$

where x' is the other intersection of ξ with C and ξ' is the other tangent to D through x', as depicted in Figure 1.2.†

Let $\eta = \tau \circ \sigma$ be the composition of τ and σ. Then $\eta(x, \xi) = (x', \xi')$ is a bijection of $\mathcal{M}$. Poncelet's theorem can be reformulated as follows.

Poncelet's Theorem. *If η^n has a fixed point, then η^n is the identity map on $\mathcal{M}$.*

That the above statement is a reformulation of Poncelet's theorem follows by observing that when the vertex x of the circumscribed polygon is contained in the side ξ, then x, x' are successive vertices and ξ, ξ' are successive sides of the polygon.

The proofs of the Poncelet and Cayley theorems given in this book depend on some detailed information concerning $\mathcal{M}$ and η, which we named, respectively, the Poncelet correspondence and Poncelet map associated with C and D. The proofs use some of the theory of elliptic curves, which is presented in Chapter 8.

We give a brief outline of the proofs. One first shows that $\mathcal{M}$ is an elliptic curve and, as such, $\mathcal{M}$ is endowed with a group structure.

†Figure 1.2 serves mainly as a pedagogical device, since the points and lines lie in the complex projective plane and hence cannot in general be visualized. This comment applies as well to figures in this book which depict subsets of the complex projective plane.

It is then shown that the map η is a translation of $\mathcal{M}$ for the group structure. This means that, if the group operation is written in the additive notation, then

$$\eta(p) = p + b, \quad \text{for all } p \text{ and some } b \text{ in } \mathcal{M}. \tag{1.1}$$

Repeated iteration of (1.1) gives

$$\eta^n(p) = p + nb, \quad \text{for all } p \text{ and some } b \text{ in } \mathcal{M}. \tag{1.2}$$

Thus η^n is also a translation of $\mathcal{M}$. Suppose that η^n fixes the point p_0. Letting $p = p_0$ in (1.2), we get $nb = \mathfrak{o}$, where $\mathfrak{o}$ is the zero element of $\mathcal{M}$. We conclude from (1.2) that η^n is the identity map on $\mathcal{M}$, thus proving Poncelet's theorem.

The above proof shows that the condition that η^n be the identity map is equivalent to $nb = \mathfrak{o}$. The element b is called a division point of order n on $\mathcal{M}$. Cayley's theorem is derived by relating the order of division points on an elliptic curve to properties of elliptic functions, which form the function field on the elliptic curve.

1.2. The Poncelet and Steiner Theorems—A Misleading Analogy

The illustration of Poncelet's theorem in Figure 1.1 will be referred to as the real case of the theorem. That is, the real case involves two ellipses, with one inside the other. The real case bears a resemblance to another classical geometrical theorem of Steiner, which we now state.

Steiner's Theorem. *Let C and D be two circles, with D inside C. Draw a circle Γ_0 tangent to both C and D. Next, draw a circle Γ_1 tangent to C, D, and Γ_0. Repeating the construction n times, we obtain a sequence of circles $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$ with Γ_k tangent to C, D, and Γ_{k-1}, for $1 \le k \le n$. If it happens that $\Gamma_n = \Gamma_0$, then this will also happen for all initial choices of Γ_0.*

The sequence of circles $\Gamma_0, \Gamma_1, \ldots$ is illustrated in Figure 1.3.

When C and D are two concentric circles, the Poncelet and Steiner theorems are both evident. In this case, the polygon in Poncelet's theorem and the chain of circles in Steiner's theorem can both

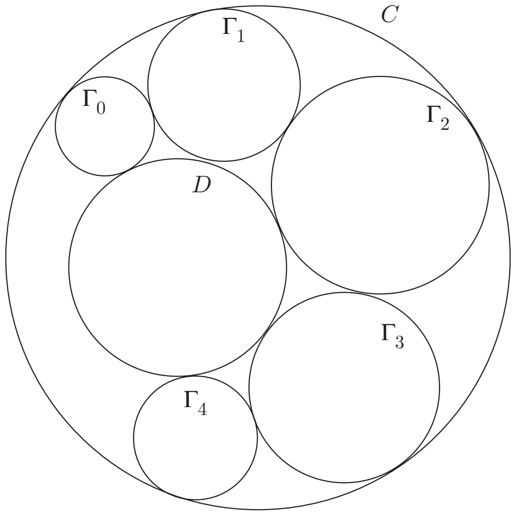

Figure 1.3

be rotated within the circular ring bounded by C and D. This raises the question of whether both theorems can be reduced to the case of concentric circles. This is the case for Steiner's theorem, but not so for Poncelet's theorem. The two theorems resemble each other in statement only, but they have very different proofs.

To reduce Steiner's theorem to the case of concentric circles, we consider transformations of the plane that take circles to circles. Transformations accomplishing this are the fractional linear (Möbius) transformations given by $w = \frac{az+b}{cz+d}$, where a, b, c, d are complex numbers satisfying $ad - bc \neq 0$. It can be shown that there exists a Möbius transformation mapping C, D to a pair of concentric circles. We omit the proof, which is given in [Sc1, p. 188].

One cannot prove Poncelet's theorem by a similar reduction. We limit our discussion to the real projective plane and consider transformations of the projective plan that take straight lines to straight lines and conics to conics. The transformations accomplishing this are ([S], Theorem 7) the projectivities

$$x_i' = \sum_{j=0}^{2} a_{ij} x_j, \quad 0 \le i \le 2_i, \ \det[a_{ij}] \ne 0.$$

We now show that when C_1, D_1 are a pair of concentric circles and C_2, D_2 are a pair of non-concentric circles (with D_i inside C_i for $i = 1, 2$), then there is no projectivity mapping one pair to the other. To prove this, we compute the number of intersecting points (in the complex projective plane) of the circles in each pair and find that these numbers are distinct. Since projectivities preserve the number of intersecting points, we conclude that there is no projectivity of the desired kind.

Let x, y, z be homogeneous coordinates and $X = x/z$, $Y = y/z$ the corresponding affine coordinates. Without loss of generality, we may assume that C_1 and C_2 are both the unit circle. Performing a rotation about the origin, we may further assume that the center of D_2 lies on the positive x-axis. The affine equations of C_1 and D_1 are

$$X^2 + Y^2 = 1, \qquad X^2 + Y^2 = r^2, \quad \text{where } 0 < r < 1,$$

and those of C_2 and D_2 are

$$X^2 + Y^2 = 1, \qquad (X - a)^2 + Y^2 = s^2, \quad \text{where } 0 < a < a + s < 1.$$

The pairs C_1, D_1 and C_2, D_2 do not intersect in the real affine plane but do intersect in the complex projective plane. Passing to homogeneous coordinates, the equations of C_1, D_1 become

$$x^2 + y^2 - z^2 = 0, \qquad x^2 + y^2 - r^2 z^2 = 0$$

and those of C_2, D_2 become

$$x^2 + y^2 - z^2 = 0, \qquad x^2 + y^2 + (a^2 - s^2) z^2 - 2axz = 0.$$

Simple computations show that the intersecting points of C_1, D_1 are $(1, \pm i, 0)$ and those of C_2, D_2 are $(1, \pm i, 0)$, $(b, \pm i\sqrt{b^2 - 1}, 1)$, where $b = \frac{1 + a^2 - s^2}{2a} > 1$. The intersections $C_1 \cap D_1$ and $C_2 \cap D_2$ thus have respective cardinalities 2 and 4, so there is no projectivity mapping one pair to the other.

1.3. The Real Case of Poncelet's Theorem

Bertrand [Sc1] has given a proof of the real case of Poncelet's theorem which does not use complex variable theory. The proof is based on notions from dynamical systems and is given in Chapter 12. We give a brief discussion here.

The main idea is to define a homeomorphism T of the outer ellipse C which is related to the Poncelet theorem and to obtain an invariant measure for it. The homeomorphism T is defined as follows. For $x \in C$, choose one of the two lines through x which is tangent to the inner ellipse D. The chosen line is required to vary continuously with x. The line meets C again in a point distinct from x, and we name this point $T(x)$. The map $x \to T(x)$ is the desired homeomorphism and is depicted in Figure 1.4.

Let T^n denote the n^{th} iterate of T. For $x \in C$, the sequence x, $T(x), \ldots, T^n(x), \ldots$ is called the T-orbit of x. Poncelet's theorem has the following reformulation in terms of T.

Poncelet's Theorem. *If the T-orbit of a point x of C closes after n steps, that is, $T^n(x) = x$, then the T-orbits of all points of C close after n steps.*

For C and D two circles, Bertrand derives, by elementary geometry, a formula for a T-invariant measure m. A measure m is said to be T-invariant if $m(T(E)) = m(E)$ for all measurable subsets of C.

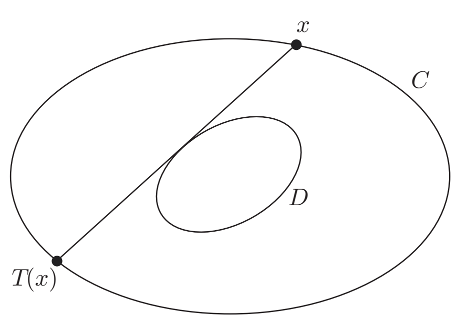

Figure 1.4

In Chapter 12 we show that Bertrand's result can be extended to the general case of two ellipses C and D.

The formula for m shows that is has the following properties: m is finite, non-atomic (points have zero measure), and assigns positive measure to arcs of positive length. In Chapter 12, we show that the existence of a T-invariant measure with these properties is equivalent to

T is topologically conjugate to a rotation.

This means that there exists a rotation ρ of the unit circle K and a homeomorphism ψ from C to K such that $T = \psi^{-1} \circ \rho \circ \psi$. In words, ρ is obtained from T via the change of variables $y = \psi(x)$.

By the conjugacy, Poncelet's theorem reduces to a corresponding statement for the rotation ρ. The latter statement is obvious, as ρ^n is also a rotation, and a rotation with a fixed point must be the identity.

We also apply the invariant measure to the study of non-periodic orbits, these being the orbits that never close up. For these, the invariant measure is used to obtain the frequency count for the number of visits of the orbit to arbitrary subarcs of C.

1.4. Related Topics

We conclude the introduction with two topics related to Poncelet's theorem. The details are presented in Chapters 13 and 14.

1.4.1. Billiards in an ellipse. The billiard problem was formulated by Birkhoff [B, p. 169] in his investigations of certain dynamical systems. A point particle moves in the interior of a domain bounded by a simple closed convex curve. The point moves along a straight line with constant velocity and is perfectly reflected at the boundary; i.e., the angle of incidence equals the angle of reflection. The problem is to give a complete description of all possible trajectories.

In general, it seems impossible to give a complete solution to this problem. A rather elegant solution does however exist when the curve is an ellipse. In this case, the billiard problem relates to Poncelet's theorem. For, aside from some exceptions listed in §13.1, the billiard trajectories have the following description. In between bounces to the boundary, the particle moves along straight line segments, and each of these segments or their extensions is tangent to a unique conic confocal with the original ellipse. The notions introduced in Chapter 12 are then used to obtain further information on the billiard problem.

1.4.2. Double queues. One of the basic problems in the analysis of queueing systems is the determination of the equilibrium probabilities. These quantities give the time-invariant distribution of the queue lengths in the system.

For double queues, the equilibrium probabilities form a doubly indexed sequence $\{p_{ij}\}$, where p_{ij} denotes the probability that the two queue lengths equal i and j. For a wide class of double queues, described in [FIM], p. 2, the following facts hold. The p_{ij} satisfy a set of linear equations which convert into an equation for the generating function $P(z,w) := \sum_{i,j} p_{ij} z^i w^j$. The equation for $P(z,w)$ exhibits a relation between $P(z,0)$ and $P(0,w)$ on a Riemann surface $\mathcal{S}$ with equation $Q(z,w) = 0$, where Q is a polynomial of degree two in z and w. Thus $\mathcal{S}$ is two-sheeted over both the z- and w-spheres. Let τ and σ denote, respectively, the interchange of the z- and w-sheets. Then the Riemann surface $\mathcal{S}$ can be identified with the Poncelet correspondence $\mathcal{M}$ of §1.1, and the sheet interchanges σ, τ of $\mathcal{S}$ can be identified with the involutions σ, τ of $\mathcal{M}$.

The relation of double queues to Poncelet's theorem comes from the fact that, in certain examples, $P(z,0)$ and $P(0,w)$ become algebraic when $\eta = \tau \circ \sigma$ is of finite order. This fact can be exploited to produce closed form solutions for $P(z,w)$, which in turn give information on the equilibrium probabilities p_{ij}.