Book I
On the Straight Line

CHAPTER I

On Angles

10. The figure formed by two rays issuing from the same endpoint is called an *angle*. The point is called the *vertex* of the angle, and the two rays are its *sides*.

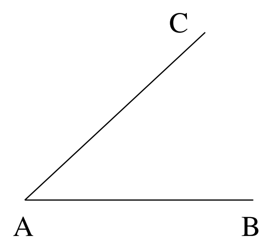

FIGURE 7

We denote an angle by the letter of its vertex, placed between two other letters which indicate the sides, often surmounted by a special symbol. If, however, the figure contains only one angle with the vertex considered, that letter will suffice to designate the angle. Thus, the angle formed by the two rays AB, AC (Fig. 7) will be denoted by $\widehat{BAC}$ or, more simply, by $\widehat{A}$.

Two angles are said to be *congruent*, in agreement with the definition of congruent figures (**3**) if they can be superimposed by a rigid motion.[1]

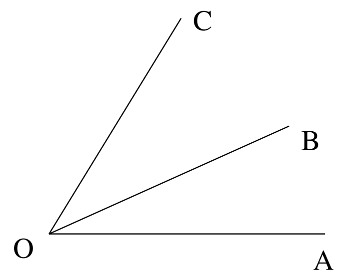

FIGURE 8

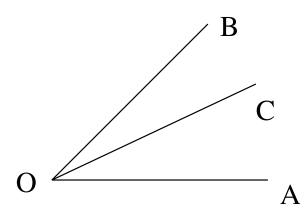

FIGURE 9

Two congruent angles, $\widehat{BAC}$, $\widehat{B'A'C'}$, can be superimposed in two different ways: namely, either AB takes the direction of $A'B'$ and AC the direction of $A'C'$, or the other way around. We pass from one to the other by turning the angle around onto itself, for instance, by moving the angle $\widehat{BAC}$ in such a way that AC occupies the position previously occupied by AB, and vice-versa.

[1]But see the footnote to **3**.–transl.

11. We say that two angles are *adjacent* if they have the same vertex, a common side, and so that they are located on opposite sides of this common side.

When angles $\widehat{AOB}$, $\widehat{BOC}$ are adjacent (Fig. 8), angle $\widehat{AOC}$ is called the *sum* of the two angles.

The sum of two or more angles is independent of the order of the angles added.

To compare angles, we move them so they have a common vertex, and a common side, and lie on the same side of this common side. Assume that $\widehat{AOB}$, $\widehat{AOC}$ are placed in this manner. If, rotating around point O, we encounter the sides in the order OA, OB, OC (Fig. 8), angle $\widehat{AOC}$ is equal to the sum of $\widehat{AOB}$ and another angle $\widehat{BOC}$; in this case angle $\widehat{AOC}$ is said to be *larger* than $\widehat{AOB}$, and the latter is *smaller* than $\widehat{AOC}$; if, on the contrary (Fig. 9) the order is OA, OC, OB, then angle $\widehat{AOC}$ is smaller than $\widehat{AOB}$. The angle $\widehat{BOC}$ which, added to the smaller angle, yields the larger angle, is the *difference* of the two angles.

Finally, in the intermediate case in which OB coincides with OC, the two angles are congruent (see **10**).

In the interior of every angle $\widehat{BAC}$ there is a ray AM which divides this angle into two congruent parts; it is called the *bisector* of the angle. The rays contained in the angle $\widehat{BAM}$ make a smaller angle with AB than with AC; the opposite is true for rays contained in the angle $\widehat{MAC}$.

An angle is said to be the *double, triple,* etc., of another if it is the sum of two, three, etc., angles equal to this other angle. The smaller angle will then be called a *half, third,* etc., of the larger one.

Remark. *The size of an angle does not depend on the size of its sides*, which are rays (**5**), and which we must imagine as extending indefinitely.

12. As we have said, an angle is determined by two rays, such as OA, OB (Fig. 10). If we extend OA past point O to form OA', and in the same way extend OB to form OB', we obtain a new angle $\widehat{A'OB'}$.

Two angles $\widehat{AOB}$, $\widehat{A'OB'}$ *such that the sides of one are extensions of the sides of the other* are called *vertical angles*.

Theorem. *Two vertical angles are congruent.*

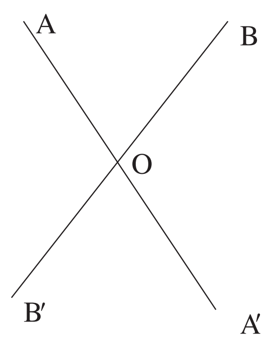

Figure 10

Indeed, let us turn (**10**) the angle $\widehat{BOA'}$ onto itself (Fig. 10). The side OB will take the place of OA', and, since $0A'$ will take the former position of OB, this means that ray OA, which is the extension of OA', will coincide with OB', the extension of OB. Thus angle $\widehat{AOB}$ is superimposed on angle $\widehat{A'OB'}$, and therefore these two angles are congruent.

13. Arcs and angles. Every ray issuing from the center of a circle meets this circle in one and only one point.

Every *angle* $\widehat{AOB}$ (Fig. 4) with its vertex at the center O of a circle determines an *arc* $\widehat{AB}$ having as endpoints the intersections of the sides of the angle with the circle. In general, (see however **20b**), we consider the arc which is less than a semicircle.

Conversely, every arc less than a semicircle can be thought of as determined by an angle with its vertex at the center of the circle, formed by the rays passing through the endpoints of the arc.

THEOREM. *On the same circle, or on equal circles*:

1°. *To equal arcs (smaller than a semicircle) there correspond equal central angles*;[2]

2°. *To unequal arcs (smaller than a semicircle) there correspond unequal central angles, and the greater angle corresponds to the greater arc*;

3°. *If an arc (smaller than a semicircle) is the sum of two others, the corresponding central angle is the sum of the angles associated with the smaller arcs.*

1°, 2°. Let $\widehat{AB}$, $\widehat{AC}$ (Fig. 4) be the two arcs, drawn on the same circle, starting from the same point A, in the same direction (**8**). The two angles $\widehat{AOB}$, $\widehat{AOC}$ are therefore placed as in **11**. But then rays OA, OB, OC are placed in the same order as points A, B, C on the circle. Moreover, if rays OB, OC coincide, then so do the points B, C, and conversely.

3°. Let us recall that the sum of two arcs (**8b**) is formed by placing them as arcs $\widehat{AB}, \widehat{BC}$ are placed (Fig. 4). Then the angle $\widehat{AOC}$ corresponding to the sum of these arcs will be the sum of $\widehat{AOB}$ and $\widehat{BOC}$, because these angles are adjacent.

According to this result, in order to compare angles, one can draw circles with the same radius, centered at the vertices of the angles, and compare the arcs intercepted on these circles.

The division of angles into two or more equal parts corresponds to the division into equal parts of the corresponding arc of a circle centered at the vertex of the angle.

14. Perpendiculars. Right angles. We say that two lines are *perpendicular* to each other if, among the four angles they form, two adjacent angles are equal to each other. For example, line AOA' (Fig. 11) is perpendicular to BOB' if the angles numbered 1 and 2 in the figure are equal. In such a case, the four angles at O are equal to each other, since angles $\widehat{3}$ and $\widehat{4}$ (Fig. 11) are equal respectively to $\widehat{1}$ and $\widehat{2}$, because they are vertical angles.

An angle whose sides are perpendicular is called a *right* angle.

[2]This is the standard term in American texts. Hadamard uses the locution 'angles with their vertex at the center' or 'angles at the center'. –transl.

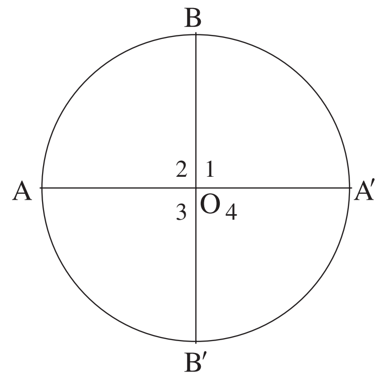

FIGURE 11

THEOREM. *In a given plane, through a given point on a line, one can draw one, and only one, perpendicular to this line.*

Suppose we want to draw a perpendicular to line AA' through point O (Fig. 11). It suffices to draw a circle with center O which cuts the line at A and A', then find point B which divides semicircle AA' into two equal parts. Then OB will be the required perpendicular; and conversely, a perpendicular to AA' drawn through point O must divide arc $\overset{\frown}{AA'}$ in half.

COROLLARY. *We see that a right angle intercepts an arc equal to one fourth of a circle centered at the vertex of the angle.*

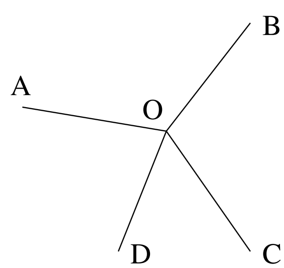

FIGURE 12

All right angles are equal, since two such angles will intercept equal arcs on equal circles centered at their vertices.

15. *If several rays are drawn issuing from the same point, the sum of the successive angles thus formed* ($\widehat{AOB}$, $\widehat{BOC}$, $\widehat{COD}$, $\widehat{DOA}$, Fig. 12) *is equal to four right angles.*

Indeed, the sum of the arcs intercepted by these angles on a circle centered at their vertex is the entire circle.

If several rays are drawn issuing from a point on a line, in such a way that all the rays are on one side of the line (Fig. 13), *the sum of the successive angles thus formed is equal to two right angles.*

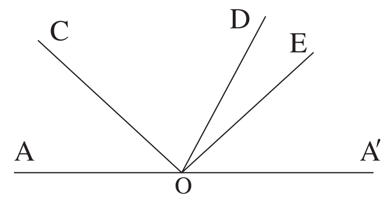

Figure 13

Indeed, the sum of the intercepted arcs is a semicircle.

Conversely, *if two or more angles with the same vertex, and each adjacent to the next, ($\widehat{AOC}$, $\widehat{COD}$, $\widehat{DOE}$, $\widehat{EOA'}$, Fig. 13) have a sum equal to two right angles, then their outer sides form a straight line.*

Indeed, these outer sides cut a circle centered at the common vertex of the angles at two diametrically opposite points, since the arc they intercept is a semicircle.

Remark. The angle AOA' (Fig. 13) whose sides form a straight line is called a *straight* angle.

15b. Theorem. *The bisectors of the four angles formed by two concurrent lines form two straight lines, perpendicular to each other.*

Let AA', BB' (Fig. 14) be two lines which intersect at O and form the angles $\widehat{AOB}$, $\widehat{BOA'}$, $\widehat{A'OB'}$, $\widehat{B'OA}$, whose bisectors are Om, On, Om', On'. We claim:

1°. That Om, Om' are collinear, as are On, On'.

2°. That the two lines thus formed are perpendicular.

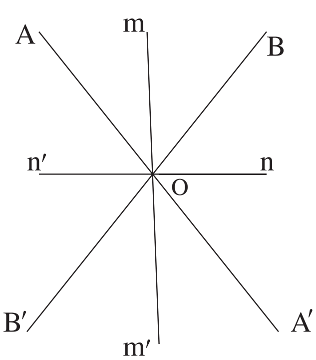

Figure 14

First, Om is perpendicular to On because, since the sum of $\widehat{AOB}$ and $\widehat{BOA'}$ is two right angles, half of each, $\widehat{mOB}$ and $\widehat{BOn}$, must add up to a right angle.

Applying the same reasoning to angles $A'OB'$ and $B'OA$, we see that Om' and On are perpendicular. Therefore Om' is the extension of Om, and likewise On' is the extension of On.

16. An angle which is less than a right angle is called an *acute* angle; an angle greater than a right angle is called an *obtuse* angle.

Two angles are said to be *complementary* if their sum is a right angle; *supplementary* if their sum is equal to two right angles.

17. Angle Measure. The *ratio* of two quantities of the same kind is[3] the number which expresses how many times one of the quantities is contained in the other. For instance, if, in dividing the segment AB into five equal parts, one of these parts is contained exactly three times in the segment BC, then the ratio of BC to AB is said to be equal to 3/5. If, on the other hand, a fifth of AB is not contained an exact number of times in BC, for example if it is contained in BC more than three times, but less than four times, then 3/5 would be an *approximate value* of the ratio $\frac{BC}{AB}$: it would be within one fifth less than the value of the ratio (the value 4/5 would be within 1/5 more).

The ratio of two quantities a, b of the same kind is equal to the ratio of two other quantities a', b' also of the same kind (but not necessarily of the same kind as the first two) if, for every n, the approximation to within $1/n$ of the first ratio is equal to the approximation within $1/n$ of the second ratio.

The *measure* of a quantity, relative to a quantity of the same kind chosen as unit, is the ratio of the given quantity to the unit.

In addition we have the following properties (*Leçons d'Arithmétique* by J. Tannery):

1°. Two quantities of the same kind, which have the same measure relative to the same unit, are equal;

2°. The ratio of two quantities of the same kind is equal to the ratio of the numbers which serve as their measures relative to the same unit;

3°. The ratio of two numbers is the same as the quotient of the two numbers; etc.

Theorem. *In the same circle, or in equal circles, the ratio of two central angles is equal to the ratio of their subtended arcs.*

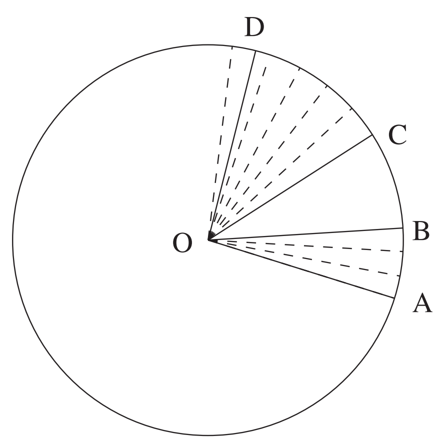

Figure 15

[3]See *Leçons d'Arithmétique* by J. Tannery, chapters X and XIII.

Consider[4] (Fig. 15) two arcs $\widehat{AB}$, $\widehat{CD}$ on circle O. Divide central angle $\widehat{COD}$ into three equal parts (for example), and suppose that one of these thirds can fit four, but not five times, into the angle $\widehat{AOB}$; the value (smaller by less than 1/3) of the ratio $\frac{\widehat{AOB}}{\widehat{COD}}$ is $\frac{4}{3}$.

But dividing the angle $\widehat{COD}$ into three equal parts, we have at the same time divided arc $\widehat{CD}$ into three equal parts (**13**). If a third of $\widehat{COD}$ fits four, but not five times, into angle $\widehat{AOB}$, it follows the same way that a third of arc $\widehat{CD}$ fits four, but not five times, into arc AB. The values to within 1/3 of the two ratios are therefore equal, and similarly the values to within $1/n$ will be equal for any integer n. The theorem is therefore proved.

Corollary. *If we take as the unit of angle measure the central angle which intercepts a unit arc, then every central angle will have the same measure as the arc contained between its sides.*

This statement is exactly the same as the preceding one, since the measure of a quantity is its ratio to the unit.

Supposing, as we will from now on, that *on each circle we choose as unit arc an arc intercepted by a unit central angle,* the preceding corollary can be stated in the abridged form: *A central angle is measued by the arc contained between its sides*.

18. The definitions reviewed above allow us to establish an important convention.

From now on, we will be able to suppose that all the quantities about which we reason have been measured with an appropriate unit chosen for each kind of quantity. Then, in all the equations we write, the quantities which appear on the two sides of an equality will represent not the quantities themselves, but rather their measures.

This will allow us to write a number of equations which otherwise would have no meaning. For instance, we might equate quantities of different kinds, since we will be dealing only with the *numbers* which measure them, the meaning of which is perfectly clear. We will also be able to consider the product of any two quantities, since we can talk about the product of two numbers, etc.

In fact, whenever we write the equality of two quantities of the same kind, this equality will have the same meaning as before, since the equality of these quantities is the same as the equality of their measures.

Following this convention we can write:

$$\widehat{AOB} = \text{arc } \widehat{\text{AB}},$$

where AB is an arc of a circle and O is the center. It is important however to insist on the fact that the preceding equality assumes that the unit of angle measure and the unit of arc measure have been chosen in the manner specified above.

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[4]This theorem becomes obvious if we consider the following proposition from arithmetic (Tannery, *Leçons d'Arithmétique*: *Two quantities are proportional if:* 1° *To any value of the first, there corresponds an equal value of the second, and* 2° *the sum of two values of the first corresponds to the sum of the corresponding values of the second*. These two conditions hold here (**13**).

The argument in the text does no more than reproduce, for this particular case, the proof of this general theorem of arithmetic.

18b. Traditionally, the circle has been divided into 360 equal parts called *degrees*, each of which contains 60 *minutes*, which are themselves divided into 60 *seconds*. One can then measure arcs in degrees and, correspondingly, angles will also be measured in degrees, and the number of degrees, minutes, and seconds of the angle will be the same as the number of degrees, minutes, and seconds of the arc intercepted by this angle on a circle centered at its vertex. A right angle corresponds to one fourth of a circle; that is, to 90 degrees. It follows that the measure of an angle at the center of a circle does not depend on the radius of the circle on which one measure the arcs, since the chosen unit of angle measure (the degree) has a value independent of this radius, namely one ninetieth of a right angle.

In writing angles (or arcs) in degrees, minutes, and seconds, we use an abridged notation: an angle of 87 degrees, 34 minutes, and 25 seconds is written: $87°34'25''$.

The introduction of decimal notation in all other kinds of measurements has led to the use of another mode of division, in which the circle is divided into 400 equal parts called *gradients* (or *grads*). The grad, a little smaller than the degree, is, as we see, one hundredth of a right angle. The grad is divided decimally, so there is actually no need for special names for its parts, which are written using the rules of decimal notation. Thus we can speak of the angle $3^G.5417$ (that is, 3 grads and 5417 ten-thousandths).

Nonetheless, a hundredth of a grad is often called a *centesimal minute*, and is indicated by the symbol ` (to distinguish it from the *hexagesimal* minute, which is a sixtieth of a degree). Likewise, one hundredth of a centesimal minute is called a *centesimal second*, denoted by the symbol ``. The angle just considered could then be written as $3^G.54$\`17\`\`.

A grad is equal to $\frac{360}{400}^{\circ}$, that is $\frac{9}{10}$ of a degree, or $54'$. A degree equals $\frac{400}{360}^{G} = 1^G.11$\`11\`\`1... (in other words, $\frac{10}{9}$ of a grad).

19. THEOREM. *Given a line, and a point not on the line, there is one, and only one, perpendicular to the line passing through the given point.*

1°. *There is one perpendicular.* Consider point O and line xy (Fig. 16). Using xy as a hinge, let us turn the half-plane containing O until it overlaps the other half-plane. Supposing point O falls on O', we join OO'. This line intersects xy because it joins two points in different half-planes. If I is the point of intersection, the angles $\widehat{O'Ix}$ and $\widehat{OIx}$ are equal because one of them can be superimposed on the other by rotation around xy. Thus the lines xy and OO' are perpendicular.

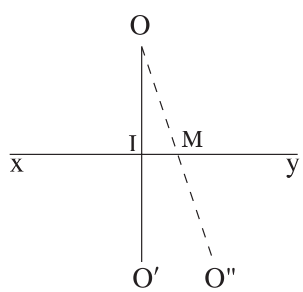

FIGURE 16

2°. *There is only one perpendicular*. Indeed, assume that OM is a perpendicular to xy passing through O. Extend the segment OM by its own length to MO''.

Again turning the half-plane containing O around xy, the segment MO will fall on MO'' because the angles $\widehat{OMx}$ and $\widehat{O''Mx}$ are right angles, and hence are equal. Since $MO'' = MO$, the point O falls on O'', and therefore O' and O'' coincide, as do the lines OO' and OO''. QED

19b. The *reflection* of a point O in a line xy is the endpoint of the perpendicular from O to the line, extended by the length of a segment equal to itself. It follows from the preceding considerations that this reflection is none other than the new position occupied by O after a rotation around xy.

Given an arbitrary figure, we can take the reflection of each of its points. The set of these reflections constitutes a new figure, called the reflection of the first. We see that in order to obtain the reflection of a given figure in line xy, we can turn the plane of the figure around xy, so that each half-plane determined by the line falls on the other, then note the position taken by the original figure. It follows that:

THEOREM. *A plane figure is congruent to its reflection.*

COROLLARY. *The reflection of a line is a line.*

When a figure coincides with its reflection in line xy, we say that it is symmetric in this line, or that this line is an *axis of symmetry* of the figure.

20. To make a figure F coincide with its reflection F', we had to use a motion which took the figure out of its plane. We must note that this superposition is not possible without such a movement; this holds because the *sense of rotation* is reversed in the two figures. We will now explain what this means.

First, let us remark that the plane of the figure divides space into two regions. For brevity, let us call one of these the region situated above the plane; the other, the region below the plane.

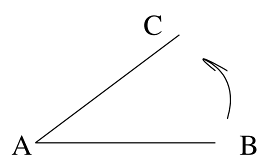

FIGURE 17

Consider now an angle $\widehat{BAC}$ in the figure F, which can be viewed as being described by a ray moving inside this angle from the position AB to the position AC (Fig. 17). *Viewed from above* this angle $\widehat{BAC}$ will be said to have an *inverse* sense of rotation or a *direct* sense, according as the moving ray turns clockwise or counterclockwise.[5] To be definite, let us consider the second situation. In this case, an observer lying along AB, with his feet at A, his head in the direction of B, and looking down, will see the side AC on his left; and therefore, if still lying on AB, he faces AC, the region below the plane will be on his right.

It is clear that to describe the sense of rotation of an angle *viewed from below*, this discussion can be repeated, with the word *above* replaced by the word *below*, and vice-versa.

[5]We note that in order to define the sense of rotation, we must take into account the order of the sides. Thus angle $\widehat{BAC}$ has the opposite sense from $\widehat{CAB}$.

Since an observer lying on AB and facing AC will necessarily have the region above the plane on his left, if the region below is on his right, and vice-versa, we see that *the sense of rotation changes depending on whether we view the angle from one side of the plane or from the other.*[6]

Let us now suppose that an angle is moved in any way at all, without ever leaving the plane. An observer moving along with the angle will not change his position relative to the regions of space above and below the plane, and therefore *the sense of rotation is not altered by a motion which does not leave the plane.*

To prove that such a motion cannot make a figure F coincide with its reflection F', it suffices therefore to see that the senses of rotation of the two figures are opposite. But we have seen that we can make F coincide with F' by turning the plane around xy (**19**). As a result of this rotation, the points above the plane are moved below it, and vice-versa. The sense of rotation of an angle in F, viewed from above, is therefore the same as the sense of F' viewed from below, so that the two angles, viewed from the same side, have opposite senses of rotation. QED

20b. Remarks. I. We say that a plane is *oriented* when a sense of rotation for angles has been chosen as the direct sense. According to the preceding paragraph, orienting a plane amounts to deciding which region of space will be said to be *above* the plane.

II. It is convenient to regard an angle with vertex O as being described by a ray starting from O which at first coincides with the first side, and then turns about O in the plane (in the direct or inverse sense) until it coincides with the second side. If the ray has made a quarter of a complete turn, the angle is a right angle. If it has made one half of a complete turn, the angle is a straight angle, such as $\widehat{AOA'}$ (Figures 11 or 13).

Nothing prevents us, by the way, from considering angles greater than two right angles, since our ray can make more than half a complete turn.

III. Clearly, an arc of a circle, like an angle, can have a direct or an inverse sense, which depends, of course, on the order in which one gives its endpoints.

Two points A, B on a circle divide it into two different arcs of which (unless the two points are diametrically opposite) one is a *minor* arc (less than a semicircle), and the other a *major* arc. It must be noted that, since the endpoints A, B are given in a certain order (for example if A is first and B is second), these two arcs have opposite senses.

Exercises

Exercise 1. Given a segment AB and its midpoint M, show that the distance CM is one half the difference between CA and CB if C is a point on the segment. If C is on line AB, but not between A and B, then CM is one half the sum of CA and CB.

Exercise 2. Given an angle $\widehat{AOB}$, and its bisector OM, show that angle $\widehat{COM}$ is one half the difference of $\widehat{COA}$ and $\widehat{COB}$ if ray OC is inside angle $\widehat{AOB}$; it is the supplement of half the difference if ray OC is inside angle $\widehat{A'OB'}$ which is vertical to $\widehat{AOB}$; it is one half the sum of $\widehat{COA}$ and $\widehat{COB}$ if OM is inside one of the other angles $\widehat{AOA'}$ and $\widehat{BOB'}$ formed by these lines.

[6] For this reason, writing viewed through a transparent sheet of paper appears reversed.

Exercise 3. Four rays OA, OB, OC, OD issue from O (in the order listed) such that $\widehat{AOB} = \widehat{COD}$ and $\widehat{BOC} = \widehat{DOA}$. Show that OA and OC are collinear, as are OB and OD.

Exercise 4. If four consecutive rays OA, OB, OC, OD are such that the bisectors of angles $\widehat{AOB}$, $\widehat{COD}$ are collinear, as are the bisectors of $\widehat{BOC}$, $\widehat{AOD}$, then these rays are collinear in pairs.

CHAPTER II

On Triangles

21. A plane region bounded by line segments is called a *polygon* (Fig 18). The segments are called the *sides* of the polygon, and their endpoints are called its *vertices*.

In fact, we will generally use the term *polygon* only for a portion of the plane bounded by a single contour which can be drawn with a single continuous stroke. The plane region which is shaded in Figure 19 will not be a polygon for us.

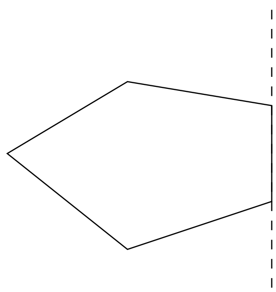

FIGURE 18

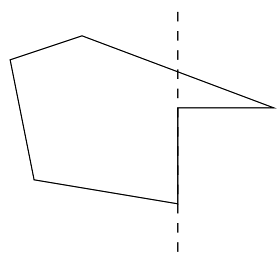

FIGURE 18b

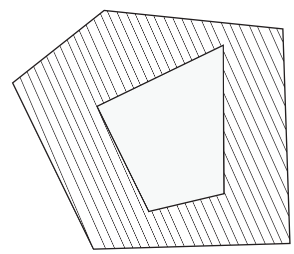

FIGURE 19

A polygon is said to be *convex* (Fig. 18) if, when we extend any side indefinitely, none of the lines thus formed crosses the polygon. In the opposite case (Fig. 18b) the polygon is said to be *concave*.

Polygons are classified according to the number of sides they have. Thus the simplest polygons are: the polygon with three sides or *triangle*, the polygon with 4 sides or *quadrilateral*, the polygon with 5 sides, or *pentagon*, the polygon with six sides or *hexagon*. We will also consider polygons with 8, 10, 12, and 15 sides, called *octagons, decagons, dodecagons, and pentadecagons*.

Any segment joining two non-consecutive vertices of a polygon is called a *diagonal*.

REMARK. More generally, an arbitrary broken line, even one whose sides cross (as in Fig. 19b), will sometimes be called a *polygon*. In this case, when the broken line does not bound a unique region, the polygon is said to be *improper*. However, when we want to specify that the polygon is of the first type (Figures 18 and 18b), and not like the one in Figure 19b, we will speak of a *proper* polygon.

22. Among triangles we distinguish in particular:

The *isosceles* triangle. This name is used for a triangle with two equal sides. The common vertex of the equal sides is called *the vertex* of the isosceles triangle, and the side opposite the vertex is called *the base*;

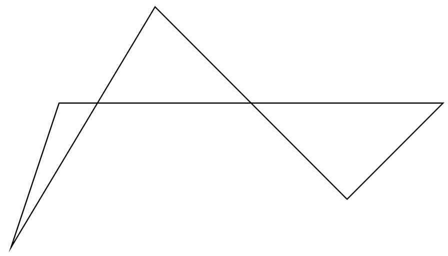

FIGURE 19b

The *equilateral* triangle, which has all three sides equal;

The *right* triangle, which has a right angle. The side opposite the right angle is called the *hypotenuse*.[1]

A perpendicular dropped from a vertex of a triangle onto the opposite side is called an *altitude* of the triangle; a *median* is a line which joins a vertex with the midpoint of the opposite side.

23. THEOREM. *In an isosceles triangle, the angles opposite the equal sides are equal.*

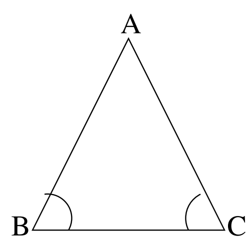

FIGURE 20

Suppose ABC is an isosceles triangle (Fig. 20). Let us turn angle $\widehat{BAC}$ onto itself (**10**) so that AB lies along the line of AC and vice-versa. Since AB and AC are equal, point B will fall on C, and C on B. Angle $\widehat{ABC}$ will therefore fall on $\widehat{ACB}$, so that these angles are equal. QED

CONVERSE. *If two angles of a triangle are equal, the triangle is isosceles.*

In triangle ABC, suppose $\widehat{B} = \widehat{C}$. Let us move this triangle so that side BC is turned around, switching points B and C. Since $\widehat{ABC} = \widehat{ACB}$, side BA will assume the direction of of CA, and conversely. Point A, which is the intersection of BA and CA, will therefore retain its original position, so that segment AB falls on segment AC. QED

COROLLARY. *An equilateral triangle is also equiangular (that is, its three angles are equal), and conversely.*

[1]The original text does not use term "leg" for a side of a right triangle that is not the hypotenuse. However, we will use this standard English term freely in the translation. –transl.

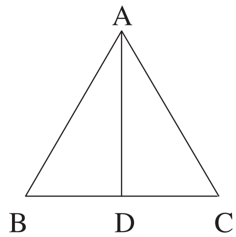

FIGURE 21

THEOREM. *In an isosceles triangle, the bisector of the angle at the vertex is perpendicular to the base, and divides it into two equal parts.*

In isosceles triangle ABC (Fig. 21), let AD be the bisector of $\widehat{A}$. In turning angle $\widehat{BAC}$ around on itself, this bisector does not move, and therefore neither does the point D where this bisector cuts the base. Segment DB falls on DC and angle $\widehat{ADB}$ on $\widehat{ADC}$. Therefore $DB = DC$ and $\widehat{ADB} = \widehat{ADC}$. QED

REMARK. In triangle ABC we can consider:

1°. The bisector of $\widehat{A}$;
2°. The altitude from A;
3°. The median from the same point;
4°. The perpendicular to the midpoint of BC.

In general, these four lines are distinct from each other (see Exercise 17). The preceding theorem shows that, in an isosceles triangle, all of these are the same line, *which is a line of symmetry of the triangle* (**19b**).

This theorem can thus be restated: *the altitude of an isosceles triangle is also an angle bisector and a median*; or *the median of an isosceles triangle is at the same time an angle bisector and an altitude; the perpendicular bisector of the base passes through the vertex and bisects the angle at the vertex.*

COROLLARY. *In an isosceles triangle, the altitudes dropped from the endpoints of the base are equal; the same is true about the medians from the endpoints of the base, about the bisectors of the angles at these points,* etc., because these segments are symmetric to each other.

24. The following propositions, known under the name of **cases of congruence of triangles**, give necessary and sufficient conditions for two triangles to be congruent.

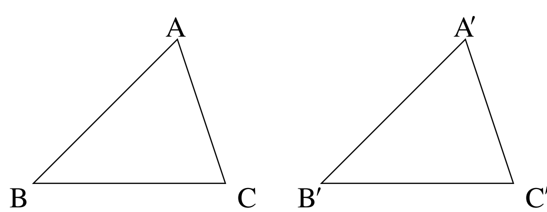

FIGURE 22

1^{st} **case: ASA.** *Two triangles are congruent if they have an equal side contained between corresponding equal angles.*

Suppose the triangles are ABC, $A'B'C'$ (*fig.* 22) in which $BC = B'C'$, $\widehat{B} = \widehat{B'}$, and $\widehat{C} = \widehat{C'}$. Let us move angle $\widehat{B'}$ onto angle $\widehat{B}$ so that side $B'A'$ lies along the line of BA and $B'C'$ along the line of BC. Since $BC = B'C'$, the point C' falls on C. Now, since $\widehat{C'} = \widehat{C}$, side $C'A'$ assumes the direction of CA, and therefore point A' falls on the intersection of BA and CA; that is, on A. This establishes the congruence of the two figures.

2^{nd} **case: SAS.** *Two triangles are congruent if they have an equal angle contained between corresponding equal sides.*

Suppose the two triangles are ABC, $A'B'C'$ (Fig. 22) in which $\widehat{A} = \widehat{A'}$, $AB = AC$, and $AC = A'C'$.

Let us move angle $\widehat{A'}$ over angle $\widehat{A}$ so that $A'B'$ assumes the direction of AB and $A'C'$ assumes the direction of AC. Since $A'B' = AB$, point B' will fall on B, and likewise C' on C. Therefore side $B'C'$ coincides with BC, and the superposition of the two figures is complete.

3^{rd} **case: SSS.** *Two triangles are congruent if they have three equal corresponding sides.*

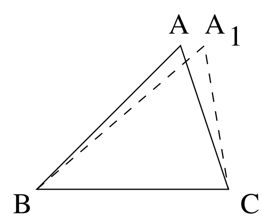

FIGURE 23

Consider triangles ABC and $A'B'C'$ such that $AB = A'B'$, $AC = A'C'$, and $BC = B'C'$, and let us move the second triangle so that side $B'C'$ coincides with BC, and the two triangles are on the same side of line BC. Denote the new position of A' by A_1. We claim that point A_1 coincides with A. This is obvious if BA_1 has the same direction as BA or if CA_1 has the same direction as CA. If this were not the case, we would have formed isosceles triangles BAA_1 and CAA_1 (Fig. 23) and the perpendicular bisector of AA_1 would have to pass through B and C (**23**, Corollary); in other words, this bisector would have to be line BC. This however is not possible, because points A and A_1 are on the same side of BC, so that BC cannot pass through the midpoint of AA_1. Thus the only possibility is that points A and A_1 must be the same point; in other words, the given triangles coincide.

REMARKS. I. In order to establish that $A = A_1$, we have investigated what would happen if these two points were distinct. Arriving, in this case, at a conclusion which is clearly false, we concluded that this possibility does not arise. This is a very useful method of reasoning, called proof *by contradiction*.

II. In a triangle there are six main elements to consider; namely, the three angles and the three sides. We have seen that if we establish the equality of three of these elements (properly chosen) in two triangles, we can conclude that the two triangles are congruent and, in particular, that the remaining elements are equal as well.

III. Two congruent triangles (or, more generally, polygons) may differ in their sense of rotation (**20**). In this case, they can only be superimposed by a motion outside the plane. On the other hand, if the sense of rotation is the same, the two polygons can be superimposed by moving them within the plane, as we will have occasion to see later on.

25. The angle formed by a side of a convex polygon and the extension of the next side is called an *exterior angle* of the polygon.

THEOREM. *An exterior angle of a triangle is greater than either of the non-adjacent interior angles.*

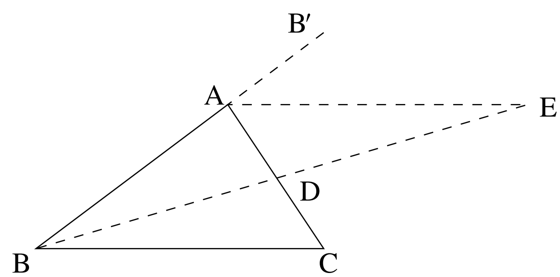

FIGURE 24

Let the triangle be ABC, and construct the exterior angle $\widehat{B'AC}$ (Fig. 24). We claim that this angle is greater, for example, than the interior angle $\widehat{C}$. To show this, we construct median BD, which we extend by its own length to a point E. The point E being inside[2] angle $\widehat{B'AC}$, this last must be greater than angle $\widehat{EAC}$.

But this last angle is precisely equal to $\widehat{C}$; indeed, triangles DAE, DBC are congruent, having an equal angle between equal sides: the angles at D are equal because they are vertical angles, and $AD = DC$, $BD = DE$ by construction. Thus exterior angle $\widehat{B'AC}$ is greater than interior angle C. QED

Exterior angle $\widehat{B'AC}$ is the supplement of interior angle $\widehat{A}$. Since angle C is smaller than the supplement of $\widehat{A}$, the sum $\widehat{A} + \widehat{C}$ is less than two right angles. Our theorem can thus be restated: *The sum of any two angles of a triangle is less than two right angles*. In particular, *a triangle cannot have more than one right or obtuse angle*.

THEOREM. *In any triangle, the greater angle lies opposite the greater side.*

In triangle ABC, suppose $AB > AC$. we will show that $\widehat{C} > \widehat{B}$. To show this, we take, on AB, a length $AD = AC$ (Fig. 25). It follows from the hypothesis that point D is between A and B, and therefore angle $\widehat{ACD}$ is less than $\widehat{C}$. But in

[2]Point E is on the same side of line BAB' as D (otherwise line DE, which has the common point B with BAB', would have to cut it again between D and E, and this is impossible), and thus on the same side as C. But B and E are on different sides of AC, which BE crosses at D.

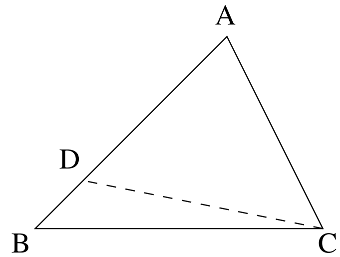

FIGURE 25

isosceles triangle ACD, $\widehat{ACD} = \widehat{ADC}$, which is greater than $\widehat{B}$ by the preceding theorem applied to triangle DCB. The theorem is thus proved.

Conversely, *the greater side corresponds*[3] *to the greater angle.*

This statement is obviously equivalent to the preceding one.

26. THEOREM. *Any side of a triangle is less than the sum of the other two.*

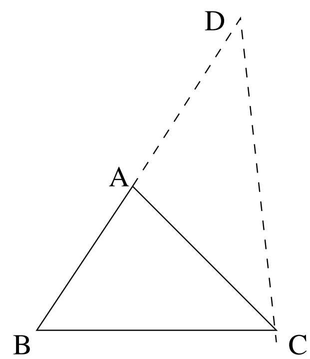

FIGURE 26

In triangle ABC, we extend side AB to a point D such that $AD = AC$ (Fig. 26). We must prove that[4] $BC < BD$. Drawing CD, we see that angle $\widehat{D}$, which is equal to angle $\widehat{ACD}$ (**23**), is therefore less than $\widehat{BCD}$.

The desired inequality thus follows from the preceding theorem applied to the triangle BCD.

COROLLARIES. I. *Any side of a triangle is greater than the difference between the other two.*

Indeed, the inequality $BC < AB + AC$ gives, after subtracting AC from both sides:

$$BC - AC < AB.$$

[3]The side *corresponding* to an angle is the side opposite it.

[4]The theorem is obvious if BC is not the largest side of the triangle.

II. *For any three points A, B, C, each of the distances AB, BC, AB is at most equal to the sum of the other two, and at least equal to the difference of the other two, equality holding if the three points are collinear.*

THEOREM. *A line segment is shorter than any broken line with the same endpoints.*

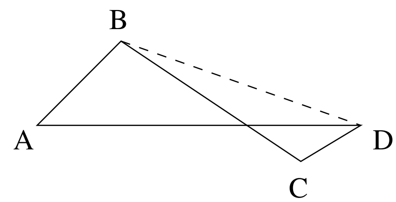

FIGURE 27

If the broken line has only two sides, the theorem reduces to the preceding one. Consider next a broken line with three sides $ABCD$ (Fig. 27). Drawing BD, we will have $AD < AB + BD$ and, since $BD < BC + CD$, we have

$$AD < AB + BD < AB + BC + CD.$$

The theorem is thus proven for a broken line with three sides. The same reasoning can be used successively for broken lines with 4 sides, 5 sides, etc. Therefore the theorem is true no matter how large the number of sides.

27. The sum of the sides of a polygon, or of a broken line, is called its *perimeter*.

THEOREM. *The perimeter of a convex broken line is less than that of any broken line with the same endpoints which surrounds it.*

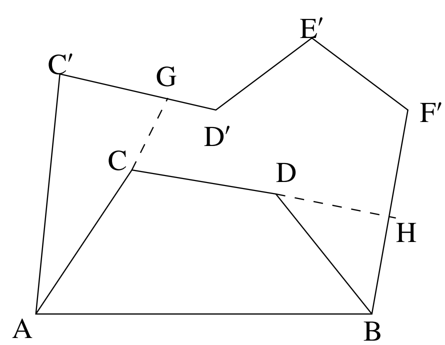

FIGURE 28

Consider the convex line $ACDB$ and the surrounding line $AC'D'E'F'B$ (Fig. 28). We extend sides AC, CD in the sense indicated by $ACDB$; that is, the side AC past C and the side CD past D. These extensions do not intersect the interior of polygon $ACDB$ because of its convexity. Suppose they cut the surrounding line in G and H respectively.

The path $ACDB$ is shorter than $ACHB$ because they have a common portion ACD, and the remainder BD of the first is less than the remainder DHB of the second. In turn, the path $ACHB$ is shorter than $AGD'E'F'B$ because, after removing the common parts AC, HB, the segment CH which is left is smaller than the broken line $CGD'E'F'H$. In the same way, $AGD'E'F'B$ is less than $AC'D'E'F'B$, because AG is less than $AC'G$. Thus

$$ACDB < ACHB < ACGD'E'F'B < AC'D'E'F'B.$$

QED

Corollary. *The perimeter of a convex polygon is less than the perimeter of any closed polygonal line surrounding it completely.*

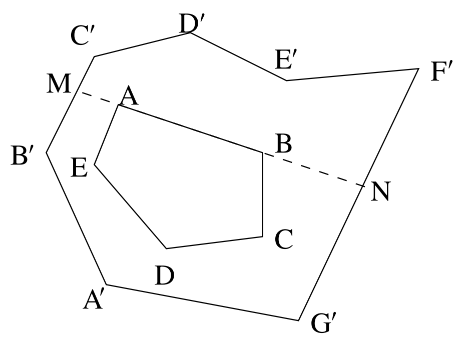

Figure 29

Consider (Fig. 29) convex polygon $ABCDE$, and polygonal line $A'B'C'D'E'\ F'G'A'$ which surrounds it completely. We extend side AB in both directions until it intersects the surrounding polygon at M, N. By the preceding result, the length of the path $AEDCB$ is less than $AMB'A'G'NB$, and therefore the perimeter of $AEDCBA$ is less than the perimeter of the polygon $NMB'A'G'N$. This perimeter, in turn, is less than the surrounding line because the part $MB'A'G'N$ is common to both, and $MN < MC'D'E'F'N$.

28. Theorem. *If two triangles have a pair of unequal angles contained by two sides equal in pairs, and the third sides of the triangles are unequal, then the greater side is opposite the greater angle.*

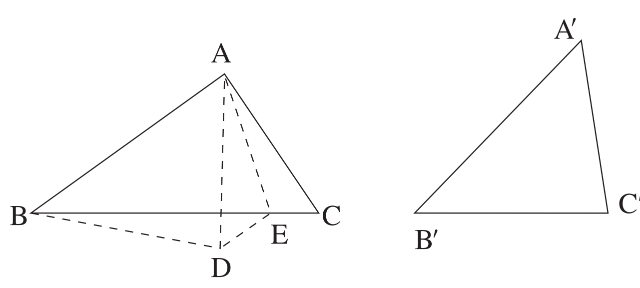

Figure 30

Consider triangles ABC, $A'B'C'$ such that $AB = A'B'$, $AC = A'C'$, and $\widehat{A} > \widehat{A'}$ (Fig. 30). We want to prove that $BC > B'C'$. Let us move the second triangle so that $A'B'$ coincides with AB. Since $\widehat{A'}$ is less than A, side $A'C'$ will move to a position AD in the interior of angle $\widehat{BAC}$. We construct the bisector AE of $\widehat{DAC}$. This segment is also inside $\widehat{BAC}$, and, as B and C lie on different sides of this line, it must intersect side BC in some point E located between B and C. If we draw DE, we see that triangles ACE and ADE are congruent, because they have an equal angle (with vertex A) between two corresponding equal sides (AE in common, $AC = A'C' = AD$). Therefore $DE = EC$. The inequality $BD < BE + ED$ provided by the triangle BDE then gives us

$$BD < BE + EC,$$

or

$$BD < BC.$$

QED

Conversely. *If, in two triangles, two pairs of sides are equal, but the third sides are unequal, then the angles opposite the unequal sides are unequal, and the greater angle is opposite the greater side.*

This statement is equivalent to the preceding one.

Remark. The preceding theorem does not require the third case of congruence for triangles (SSS). It therefore provides another proof of that case.

Indeed, if we have $AB = A'B'$, $AC = A'C'$ and, in addition $BC = B'C'$, the angles $\widehat{A}$ and $\widehat{A'}$ would have to be equal, or else BC and $B'C'$ could not be equal. Knowing now that $\widehat{A} = \widehat{A'}$ the two triangles are congruent by the second case (SAS).

Exercises

Exercise 5. Prove that a triangle is isosceles:
1°. if an angle bisector is also an altitude;
2°. if a median is also an altitude;
3°. if an angle bisector is also a median.

Exercise 6. On side Ox of some angle, we take two lengths OA, OB, and on side Ox' we take two lengths OA', OB', equal respectively to the first two lengths. We draw AB', BA', which cross each other. Show that point I, where these two segments intersect, lies on the bisector of the given angle.

Exercise 7. If two sides of a triangle are unequal, then the median between these two sides makes the greater angle with the smaller side. (Imitate the construction in **25**.)

Exercise 8. If a point in the plane of a triangle is joined to the three vertices of a triangle, then the sum of these segments is greater than the semi-perimeter of the triangle; if the point is inside the triangle, the sum is less than the whole perimeter.

Exercise 8b. If a point in the plane of a polygon is joined to the vertices of the polygon, then the sum of these segments is greater than the semi-perimeter of the polygon.

Exercise 9. The sum of the diagonals of a [convex[5]] quadrilateral is between the semi-perimeter and the whole perimeter.

Exercise 10. The intersection point of the diagonals of a [convex[6]] quadrilateral is the point in the plane such that the sum of its distances to the four vertices is as small as possible.

Exercise 11. A median of a triangle is smaller than half the sum of the sides surrounding it, and greater than the difference between this sum and half of the third side.

Exercise 12. The sum of the medians of a triangle is greater than its semi-perimeter and less than its whole perimeter.

Exercise 13. On a given line, find a point such that the sum of its distances to two given points is as small as possible. Distinguish two cases, according to whether the points are on the same side of the line or not. The second case can be reduced to the first (by reflecting part of the figure in the given line).

Exercise 14. (Billiard problem.) Given a line xy and two points A, B on the same side of the line, find a point M on this line such that $\widehat{AMx} = \widehat{BMy}$.

We obtain the same point as in the preceding problem.

Exercise 15. On a given line, find a point with the property that the difference of its distances to two given points is as large as possible. Distinguish two cases, according to whether the points are on the same side of the line or not.

[5] This condition is not in the original text. See solution. –transl.

[6] See the previous footnote. –transl.

CHAPTER III

Perpendiculars and Oblique Line Segments

29. THEOREM. *If, from a given point outside a line, we draw a perpendicular and several oblique line segments:*

1°. *The perpendicular is shorter than any oblique segment*;

2°. *Two oblique segments whose feet are equally distant from the foot of the perpendicular are equal*;

3°. *Of two oblique segments, the longer is the one whose foot is further from the foot of the perpendicular.*

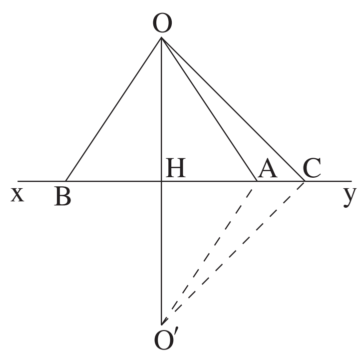

FIGURE 31

1°. Consider perpendicular OH and oblique segment OA from point O to line xy (Fig. 31). If we extend OH by a length HO' equal to itself, then O' is symmetric to O with respect to line xy. Therefore $O'A$ is the symmetric image of OA, and the two are equal. Now in triangle $OO'A$ we have

$$OO' < OA + O'A,$$

and we can replace OO' with $2OH$ and $OA + OA'$ with $2OA$. Thus we find $2OH < 2OA$, or $OH < OA$.

2°. Consider next the oblique segments OA, OB such that $HA = HB$. These two oblique segments will be equal by symmetry with respect to line OH.

3°. Finally, let OA and OC be oblique segments such that $HC > HA$ (Fig. 31). Suppose first that points A and C are on the same side of H. Then point A is inside triangle $OO'C$. By (**27**) we have

$$OA + O'A < OC + O'C$$

and, as we saw before, $OA = O'A$ and $OC = O'C$. Dividing by two, we have, as before,

$$OA < OC.$$

If we had started with an oblique segment OB less distant than OC, but on the other side of H, it would suffice to construct a length $HA = HB$ in the direction of HC. The oblique segment OB then would be equal to OA (2°), and hence smaller than OC as we have seen above.

30. Conversely. *If two oblique segments are equal, their feet are equally distant from the foot of the perpendicular,* otherwise they are unequal; *if they are unequal, the longer is the more distant from the perpendicular.*

Corollary. *There are no more than two oblique segments of the same length from the same point O to a line xy.*

This is true because the feet of these oblique segments are equally distant from H, and there are only two points on xy at a given distance from H.

31. The length of the perpendicular dropped from a point to a line is called the *distance from the point to the line*. The preceding theorem shows that this perpendicular is in fact the shortest path from the point to the line.

32. Theorem.

1°. *Every point on the perpendicular bisector of a segment is equally distant from the endpoints of the segment*;

2°. *A point not on the perpendicular bisector is not equally distant from the endpoints of the segment.*

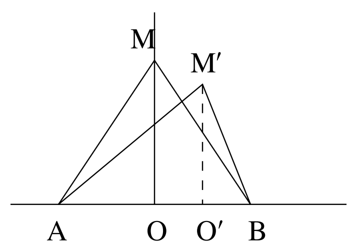

Figure 32

1°. If M is on the perpendicular bisector of AB (Fig. 32), the segments MA, MB are equal, because they are oblique lines which are equally distant from the foot of the perpendicular MO.

2°. Let M' be a point not on the perpendicular bisector, and suppose it is on the same side of the bisector as point B. The foot O' of the perpendicular dropped from M onto line AB will be on the same side of the bisector (otherwise this perpendicular would meet the bisector and, from their intersection, there would be two lines perpendicular to AB).

We will then have $O'A > O'B$ and therefore (**29**)

$$M'A > M'B.$$

QED

Remark I. We could have proved the second part in a different way by establishing the following equivalent proposition: *Any point which is equidistant from A and B is on the perpendicular bisector of AB.* This follows from the converse of **30** (the feet of two equal oblique segments are equidistant from the foot of the perpendicular). One could also use the properties of isosceles triangles (**23**, Remark). However, one must observe that our way of proceeding has the advantage of showing which of the two distances is the greater when they are unequal.

Remark II. The proposition which we just stated: *Any point which is equidistant from A and B is on the perpendicular bisector of AB,* is the converse of the first part of the preceding theorem. We have here two different ways of proving a converse. The first consists in following the original reasoning in reverse. This is what we did in the preceding remark. The original reasoning (1° in the previous theorem) started from the hypothesis that point M is on the perpendicular bisector; in other words, that the feet of MA, MB were equally distant from the perpendicular from M, and deduced from this that MA and MB are equal. This time, we started from the hypothesis that MA and MB are equal, and we concluded that their feet are equally distant.

The second method of proving the converse consists in proving what we call the *inverse* statement. This name is given to the statement whose hypothesis is the negation of the original hypothesis, and whose conclusion is the negation of the original conclusion. Thus the second part of the preceding theorem is the inverse of the first, and is equivalent to its converse.

We will find later on (see, for example, **41**) a third method of proving a converse.

33. Let us now make use of the definition of **1b**.

Using this definition, the preceding theorem can be restated as follows:

Theorem. *The locus of points equidistant from two given points is the perpendicular bisector of the segment joining these two points.*

This is true because the figure formed by the points equidistant from A and B is in fact the perpendicular bisector of AB.

We remark that, to establish this fact, one must prove, as we have, that:

1°. every point on the perpendicular bisector satisfies the given condition;

2°. every point satisfying this condition is on the perpendicular bisector; or, equivalently, that no point outside the bisector satisfies the condition. This kind of double argument is necessary in all problems about geometric loci.

Exercises

Exercise 16. If the legs of a first right triangle are respectively smaller than the legs of a second, then the hypotenuse of the first is smaller than the hypotenuse of the second.

Exercise 17. If the angles $\widehat{B}$ and $\widehat{C}$ of a triangle ABC are acute, and the sides AB, AC unequal, then the lines starting from A are, in decreasing order of length, as follows: larger side, median (see Exercise 7 in Chapter II), angle bisector, smaller side, altitude.

Exercise 18. The median of a non-isosceles triangle is greater than the bisector from the same vertex, bounded by the third side.

CHAPTER IV

Cases of Congruence for Right Triangles. A Property of the Bisector of an Angle

34. Cases of congruence for right triangles. The general cases of congruence for triangles apply, of course, to right triangles as well. For instance, two right triangles are congruent if their legs are respectively equal (2^{nd} case of congruence for any triangles).

Aside from these general cases, right triangles present two special cases of congruence.

First case of congruence. *Two right triangles are congruent if they have equal hypotenuses, and an equal acute angle.*

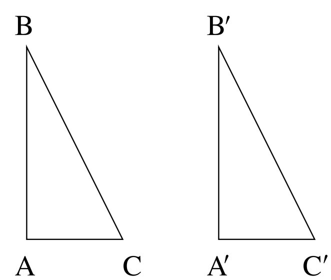

FIGURE 33

Consider two right triangles $ABC, A'B'C'$ (Fig. 33) in which $BC = B'C'$, $\widehat{B} = \widehat{B'}$. We move the second triangle over the first so that the angles B and B' coincide. Then $B'C'$ will assume the direction of BC, and, since these two segments are equal, C' will fall on C. $B'A'$ will assume the direction of BA, and therefore $C'A'$ will coincide with the perpendicular dropped from C on BA; that is, with CA.

Second case of congruence. *Two right triangles are congruent if they have equal hypotenuses and one pair of corresponding legs equal.*

Suppose the two triangles are ABC, $A'B'C'$, and that $BC = B'C'$, $AB = A'B'$. We move the second triangle onto the first so that $A'B'$ coincides with AB. The side AC will assume the direction of $A'C'$. We then have two oblique lines from point B to line AC; namely, BC and the new position of $B'C'$. These oblique lines are equal by hypothesis and therefore (**30**), equidistant from the foot of the perpendicular. This gives $A'C' = AC$, from which the congruence of the triangles follows.

35. Theorem. *If two right triangles have equal hypotenuses and an unequal acute angle, then the sides opposite the unequal angles are unequal, and the larger side is opposite the larger angle.*

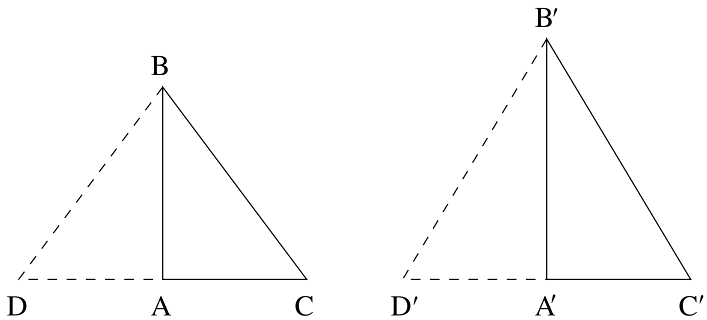

Figure 34

Let the two triangles be ABC, $A'B'C'$ (Fig. 34), for which $BC = B'C'$, $\widehat{B} > \widehat{B'}$. We claim that $AC > A'C'$.

To see this, we extend AC by its own length to get AD, and similarly extend $A'C'$ to get $A'D'$. We immediately have (**29**) $BD = BC = B'C' = B'D'$. Moreover, in isosceles triangle BDC median BA is also an angle bisector, so that angle $\widehat{DBC}$ is twice the original angle $\widehat{B}$. Likewise, angle $\widehat{D'B'C'}$ is twice the original angle $\widehat{B'}$, from which we have $\widehat{DBC} > \widehat{D'B'C'}$.

The two triangles DBC, $D'B'C'$ therefore have an unequal angle between equal sides, from which it follows that $DC > D'C'$, and therefore $AC > A'C'$.

36. Theorem. *The bisector of an angle is the locus of points in the interior of the angle which are equidistant from the two sides.*

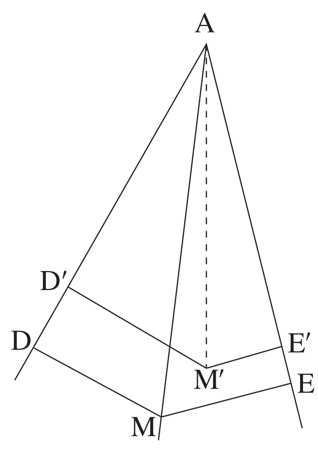

Figure 35

As explained earlier (**33**), the proof consists of two parts:

1°. *Every point on the bisector is equidistant from the two sides.*

Suppose the angle is $\widehat{ABC}$, and M is a point on its bisector. If we drop perpendiculars MD, ME to the sides of the angle, then right triangles AMD and AME are congruent, because they have a common hypotenuse, and an equal acute angle (at A) by hypothesis. Therefore the perpendiculars MD, ME are equal.

2°. *The distances from any point inside the angle, but not on its bisector, to the two sides are unequal.*

Suppose point M' lies, for instance, between the bisector and side AC. Then angle $\widehat{BAM'}$ will be greater than $\widehat{M'AC}$. If we drop perpendiculars $M'D'$, $M'E'$ onto sides AB, AC, then right triangles $AM'D'$, $AM'E'$ will have a common hypotenuse and an unequal angle at A. Therefore $M'D'$ will (**35**) be greater than $M'E'$.

As in the case of the theorem of **32**, we could have proved, in place of the inverse proposition in 2° above, the converse proposition: *Any point inside an angle, and equidistant from its two sides, is on its bisector*. To do this, following the original reasoning in reverse, we would have considered a point M equidistant from the two sides, and would have applied the second case of congruence (**34**) to the two right triangles AMD, AME, which have a common hypotenuse and in which $MD = ME$. We would have concluded that the angles at A are equal, so that AM is the angle bisector. However, we would not have established which is the larger distance when they are unequal.

Corollaries. I. *This theorem allows us to give a second* **definition** *of the bisector of an angle, namely: The bisector of an angle is the locus of the points inside the angle which are equidistant from the sides.*

We observe that this second definition is exactly equivalent to the one given in **11**.

II. *The locus of all points equidistant from two intersecting lines is composed of the two bisectors (**16**) of the angles formed by these two lines.*

Exercises

Exercise 19. Show that a triangle is isosceles if it has two equal altitudes.

Exercise 20. More generally, in any triangle, the smaller altitude corresponds to the larger side.

CHAPTER V

Parallel Lines

37. When two lines (Fig. 36) are intersected by a third (called a transversal), this last line forms eight angles with the first two, which are numbered in the figure. The relative positions of these angles are described as follows.

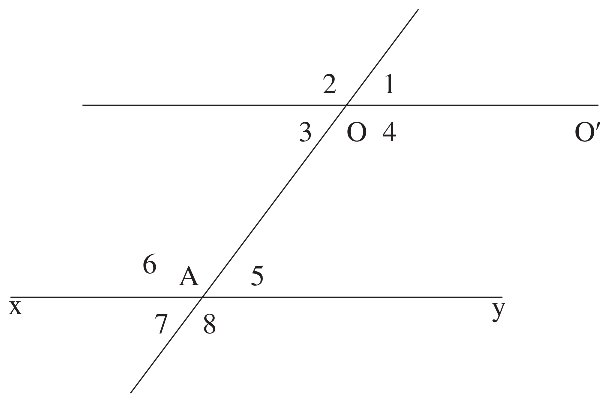

FIGURE 36

Two angles such as $\widehat{3}$ and $\widehat{5}$, situated between the two lines, and on different sides of the transversal, are called *alternate interior* angles.

Two angles such as $\widehat{3}$ and $\widehat{6}$ situated between the two lines, but on the same side of the transversal, are said to be *interior on the same side*.

Two angles such as $\widehat{6}$ and $\widehat{2}$ on the same side of the transversal, one between the two lines, one outside, are said to be *corresponding.*

38. DEFINITION. Two lines in the same plane are said to be *parallel* if they do not intersect, no matter how far extended in either direction.

THEOREM. *Two lines intersected by the same transversal are parallel*:
1°. *If the interior angles on the same side are supplementary*;[1]
2°. *If alternate interior angles are equal*;
3°. *If corresponding angles are equal.*

1°. If the two lines were to intersect, on either side of the transversal, they would form a triangle in which (**25**) the sum of two interior angles on the same side would have to be less than two right angles.

[1]If the angles $\widehat{3}$ and $\widehat{6}$ are supplementary, then so are $\widehat{4}$ and $\widehat{5}$, since the sum of these four angles is four right angles.

The other two cases can be reduced to the first:

2°. If $\widehat{3} = \widehat{5}$, this is equivalent to saying that $\widehat{3}$ is the supplement of $\widehat{6}$, or that the interior angles on the same side are supplementary.

3°. If $\widehat{6} = \widehat{2}$, then again $\widehat{3}$ and $\widehat{6}$ are supplementary, because $\widehat{3}$ is the supplement of $\widehat{2}$.

This theorem can be used to prove that two lines are parallel.

Corollary. *In particular, two lines perpendicular to a third line are parallel.*

39. Theorem. *Through any point not on a given line, a parallel to the given line can be drawn.*

Consider point O and line xy (Fig. 36). If we join point O to any point A of xy, then line OO', which makes an angle with OA such that $\widehat{AOO'} + \widehat{OAy}$ is equal to two right angles, will be parallel to xy.

40. Because the preceding construction can be made in infinitely many ways (since point A can be chosen anywhere on line xy), it would seem that there are infinitely many different parallels.

This is not true, however, if we adopt the following **axiom**:

Axiom. *Through any point not on a given line, only one parallel to the given line can be drawn.*[2]

Corollaries. I. *Two distinct lines parallel to a third line are themselves parallel; since if the two lines had a point in common, two lines would pass through it, each parallel to the third line.*

II. *If two lines are parallel, any third line which intersects one of them must intersect the other,* otherwise two parallels to the second line would intersect each other.[3]

41. The most important proposition in the theory of parallels is the following converse of the theorem in **38**.

Converse. *When two parallel lines are intersected by the same transversal*:

1°. *The interior angles on the same side are supplementary*;

2°. *The alternate interior angles are equal*;

3°. *The corresponding angles are equal.*

The proof is the same in all three cases. Consider the parallel lines AB, CD intersected by the transversal EFx (Fig. 37). We claim, for example, that the corresponding angles $\widehat{xEB}$ and $\widehat{xFD}$ are equal. Indeed, we can construct, at point E, an angle $\widehat{xEB'}$ equal to angle $\widehat{xFD}$. Line EB' will then be parallel to CD, and therefore it will coincide with EB.

Corollaries. I. *If two lines determine, with a common transversal, two interior angles on the same side which are not supplementary, then the two lines are not parallel, and they intersect on the side of the transversal where the sum of the interior angles is less than two right angles.*

II. *When two lines are parallel, any line perpendicular to one of them is perpendicular to the other.*

---

[2]This axiom is known as *Euclid's Postulate*. In fact it should be viewed as part of the definition in fundamental notions. (See Note B at the end of this volume.)

[3]We have here another example of a proof by contradiction (see **24**, Remark I.)

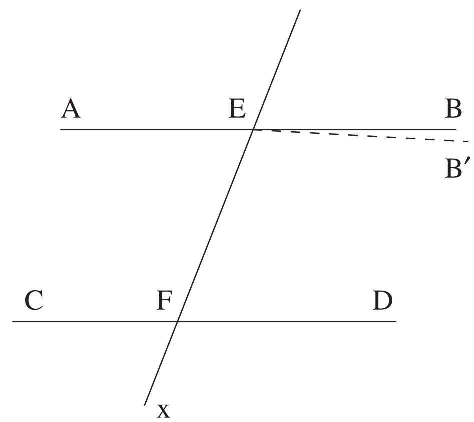

FIGURE 37

This is true because the perpendicular must intersect the other line (**40**, Corollary II) and the angle it forms will be a right angle, because of the theorem we have just proved.

REMARKS. I. The corresponding equal angles $\widehat{xEB}$ and $\widehat{xFD}$ have the same direction of rotation.

The two parallel directions EB, FD, both situated on the same side of a common transversal, are called *parallel and in the same direction.*

II. We have used a new method of proof of a converse, different from the ones used in **32**, and which consists in establishing the converse with the help of the original theorem. It should be noted here that the argument basically depends on the fact that the parallel through E to CD is *unique.*

42. According to the theorem of **38** and its converse, the definition of parallels amounts to the following: *Two lines are parallel if they form, with an arbitrary transversal, equal corresponding angles (or equal alternate interior angles, or supplementary interior angles on the same side of the transversal).*

This definition, equivalent to the first, is usually more convenient to use.

In place of the phrase *parallel lines* we often substitutes *lines with the same direction*, whose meaning is clear from the preceding propositions.

REMARK. Because of what we have just noted, *two lines which coincide must be viewed as a particular case of two parallel lines.*

43. THEOREM. *Two angles with two pairs of parallel sides are either equal or supplementary. They are equal if the sides both lie in the same direction or both in opposite directions; they are supplementary if a pair of sides lie in the same direction, and the other pair opposite.*

First, two angles with a common side, and whose second sides are parallel in the same direction (Fig. 38) are equal because they are corresponding angles. Two angles whose sides are parallel and in the same direction are then equal because one side of the first angle and one side of the second angle will form a third angle equal to the first two. If one of the sides is in the same direction, with the other

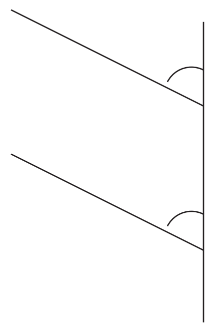

FIGURE 38

opposite, then extending the side which is in the opposite direction we obtain an angle supplementary to the first angle, and equal to the second.

If both sides are in opposite directions, we extend both sides of the first angle. Thus we form an angle equal to the first because they are vertical angles, and also equal to the second.

REMARK. Two corresponding angles and, therefore, two angles with sides parallel and in the same direction, have the same sense of rotation. We can therefore say: *Two angles with parallel sides are equal or supplementary, according as whether they have the same sense of rotation or not.*

THEOREM. *Two angles whose sides are perpendicular in pairs are equal or supplementary, according as whether they have the same sense of rotation or not.*

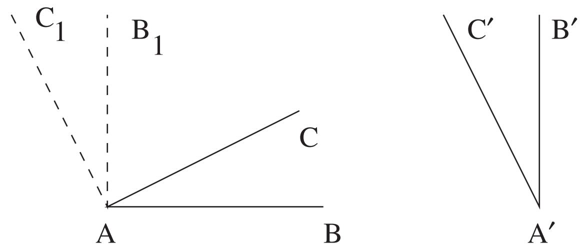

FIGURE 39

Consider angles $\widehat{BAC}$, $\widehat{B'A'C'}$ (Fig. 39) such that $A'B'$ and $A'C'$ are perpendicular respectively to AB and AC. We draw AB_1 perpendicular to AB, and turn angle $\widehat{B_1AC}$ around on itself: side AB_1 falls on AC, and hence line AB, perpendicular to AB_1, will occupy a position AC_1 perpendicular to AC. We now have an angle $\widehat{B_1AC_1}$ which is equal to $\widehat{BAC}$ and has the same sense (because it was constructed by reflection of angle $\widehat{CAB}$, whose sense is opposite to that of $\widehat{BAC}$) and whose sides, perpendicular respectively to those of the first angle, are therefore parallel to those of $\widehat{B'A'C'}$. Since the angles $\widehat{BAC}$, $\widehat{B'A'C'}$ are equal or supplementary, according as their sides have the same or opposite senses of rotation, the same is true of the given angles.

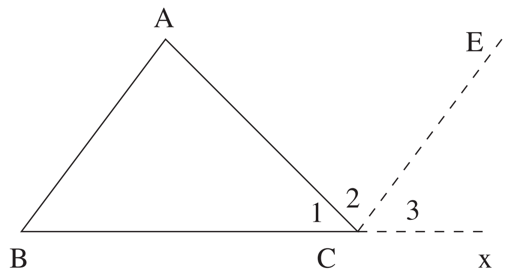

FIGURE 40

44. THEOREM. *The sum of the angles of a triangle is equal to two right angles.*

In triangle ABC (Fig. 40), we extend BC in the direction Cx, and draw CE parallel to AB. We form three angles at C (numbered 1 to 3 in the figure), whose sum is equal to two right angles. These angles are equal to the three angles of the triangle; namely: $\widehat{1}$ is the angle $\widehat{C}$ of the triangle; $\widehat{2} = \widehat{A}$ (these are alternate interior angle formed by transversal AC with the parallel lines AB, CE); $\widehat{3} = \widehat{B}$ (these are corresponding angles formed by transversal BC with the same parallels).

COROLLARIES. I. *An exterior angle of a triangle is equal to the sum of the non-adjacent interior angles.*

II. *The acute angles of a right triangles are complementary.*

III. *If two triangles have two pairs of equal angles, then the third pair of angles is equal as well.*

44b. THEOREM. *The sum of the interior angles of a convex polygon*[4] *is equal to two less than the number of sides times two right angles.*

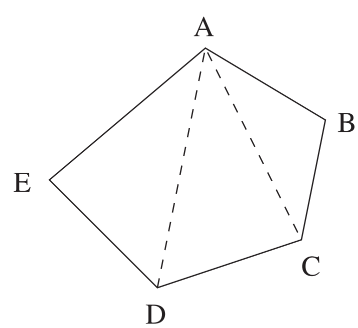

FIGURE 41

Let the polygon be $ABCDE$ (Fig. 41). Joining A to the other vertices, we decompose the polygon into triangles. The number of triangles is equal to the

[4]The theorem remains true if one takes, as the interior angle at any vertex pointing inwards (Fig. 19b), the one which is directed towards the interior of the polygon, an angle which in this case is greater than two right angles. In the case of Fig. 19b the proof would proceed as in the text, drawing the diagonals starting from the vertex pointing inwards. If there are several inward-pointing vertices, the theorem would still be true, but the proof would be more difficult.

number of sides minus two because if A is taken as the common vertex of these triangles, all the other sides of the polygon are the sides opposite A, except the two sides which end at A. The sum of the angles of these triangles gives us the sum of the angles of the polygon. The theorem is proved.

If n is the number of sides of the polygon, the sum of the angles is $2(n-2)$ or $2n-4$ right angles.

Corollary. *The sum of the exterior angles of a convex polygon, formed by extending the sides in the same sense, is equal to four right angles.*

Indeed, an interior angle plus the adjacent exterior angle gives us two right angles. Adding the results for all the n vertices we obtain $2n$ right angles, of which $2n-4$ are given by the sum of the interior angle. The sum of the exterior angles is equal to the four missing right angles.

Exercises

Exercise 21. In a triangle ABC, we draw a parallel to BC through the intersection point of the bisectors of $\widehat{B}$ and $\widehat{C}$. This parallel intersects AB in M and BC in N. Show that $MN = BM + CN$.

What happens to this statement if the parallel is drawn through the intersection point of the bisectors of the exterior angles at B and C? Or through the intersection of the bisector of $\widehat{B}$ with the bisector of the exterior angle at C?

Sum of the angles of a polygon.

Exercise 22. Prove the theorem in **44b** by decomposing the polygon into triangles using segments starting from an interior point of the polygon.

Exercise 23. In triangle ABC we draw lines AD, AE from point A to side BC, such that the first makes an angle equal to $\widehat{C}$ with AB, while the second makes an angle equal to $\widehat{B}$ with AC. Show that triangle ADE is isosceles.

Exercise 24. In any triangle ABC:

1°. The bisector of $\widehat{A}$ and the altitude from A make an angle equal to half the difference between $\widehat{B}$ and $\widehat{C}$.

2°. The bisectors of $\widehat{B}$ and $\widehat{C}$ form an angle equal to $\frac{1}{2}\widehat{A}$ + one right angle.

3°. The bisectors of the exterior angles of $\widehat{B}$ and $\widehat{C}$ form an angle equal to one right angle $-\frac{1}{2}\widehat{A}$.

25. In a convex quadrilateral:

1°. The bisectors of two consecutive angles form an angle equal to one half the sum of the other two angles.

2°. The bisectors of two opposite angles form a supplementary angle to the half the difference of the other two angles.

CHAPTER VI

On Parallelograms. — On Translations

45. Among quadrilaterals, we consider in particular *trapezoids* and *parallelograms.* A quadrilateral is called a *trapezoid* (Fig. 42) if it has two parallel sides. These parallel sides are called the *bases* of the trapezoid. A quadrilateral with two pairs of parallel sides is called a *parallelogram.* (Fig. 43).

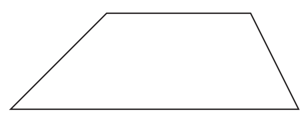

FIGURE 42

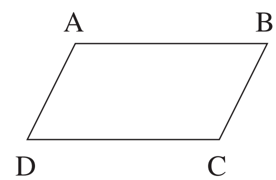

FIGURE 43

THEOREM. *In a parallelogram, opposite angles are equal, and angles adjacent to the same side are supplementary.*

Indeed, in parallelogram $ABCD$ (Fig. 43), angles $\widehat{A}$ and $\widehat{C}$, adjacent to side AB, are interior angles on the same side formed by parallels AD, BC cut by transversal AB; therefore they are supplementary. The opposite angles A, C are equal because they have parallel sides in opposite directions.

REMARK. We see that it suffices to know one angle of a parallelogram in order to know all of them.

Converse. *If, in a quadrilateral, the opposite angles are equal, then the quadrilateral is a parallelogram.*

Indeed, the sum of the four angles of a quadrilateral equals four right angles (**44b**). Since $\widehat{A} = \widehat{C}$, $\widehat{B} = \widehat{D}$, the sum of the four angles $\widehat{A} + \widehat{B} + \widehat{C} + \widehat{D}$ can be writen as $2\widehat{A} + 2\widehat{B}$. We then have $\widehat{A} + \widehat{B} =$ two right angles, so the lines AD, BC are parallel, since they form two supplementary interior angles on the same side of transversal AB. Similarly, we can show that AB is parallel to CD.

46. THEOREM. *In any parallelogram, the opposite sides are equal.*

In parallelogram $ABCD$ (Fig. 44), we draw diagonal AC. This diagonal divides the parallelogram into two triangles ABC, CDA, which are congruent because they have the common side AC between equal angles: $\widehat{A_1} = \widehat{C_1}$ are alternate interior angles formed by parallels AB, CD, and $\widehat{A_2} = \widehat{C_2}$ are alternate interior angles formed by parallels AD, BC.

These congruent triangles yield $AB = CD$, $AD = BC$.

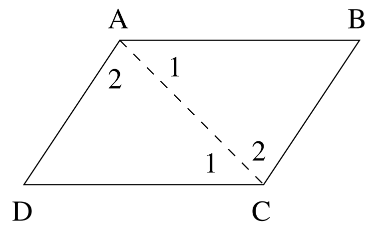

FIGURE 44

In this theorem, the hypothesis consists of two parts:
1°. two sides are parallel;
2°. the other two sides are also parallel.

The conclusion also has two parts:
1°. two opposite sides are equal;
2°. the other two sides are also equal.

Since we can form a converse by using either the whole or a part of the original conclusion, and *vice versa,* this theorem has two converses.

Converses. *A quadrilateral is a parallelogram:*
1°*. If the opposite sides are equal*;
2°*. If two opposite sides are parallel and equal.*

1°. Suppose that $AB = CD$ and $AD = BC$ in quadrilateral $ABCD$ (Fig. 44). We again draw diagonal AC. Triangles ABC, CDA will be congruent, since they have three sides equal in pairs. Thus angles $\widehat{A_1}$ and $\widehat{C_1}$ are equal, and since these are alternate interior angles with respect to transversal AC, lines AB, CD must be parallel. Likewise, the equality of $\widehat{A_2}$, $\widehat{C_2}$ proves that AD and BC are parallel.

2°. Assume now that $AB = CD$ and AB is parallel to CD. Triangles ABC, CDA are again congruent, because they have an equal angle ($\widehat{A_1} = \widehat{C_1}$ are alternate interior angles) between two pairs of equal sides: AC is a common side and $AB = CD$. From this congruence we again find that $\widehat{A_2} = \widehat{C_2}$, and that sides AD, BC are parallel.

46b. REMARK I. A quadrilateral can have two sides AB, CD equal, and the other two BC, AD parallel, without being a parallelogram (it is then called an *isosceles trapezoid*).

Choosing an arbitrary side AB, it suffices to take the line symmetric to AB (D_1C, Fig. 45) with respect to any line xy in the plane, provided that the line xy is not parallel to AB, and that xy intersects the line AB in a point I on the extension of AB (and not on segment AB itself). Then quadrilateral $ABCD_1$ will have two parallel sides (both perpendicular to xy) and the other two equal (because they are symmetric to each other). These last two sides are not parallel because they intersect at point I.

Conversely, every quadrilateral with two sides BC, AD parallel, and the other two equal is either a parallelogram, or (in the case of an isosceles trapezoid) a figure with a line of symmetry. Indeed, let xy be the perpendicular bisector of BC. We have already found two oblique lines from C to the line AD, both equal to AB, namely: CD_1 which is symmetric to AB with respect to xy, and CD_2 which,

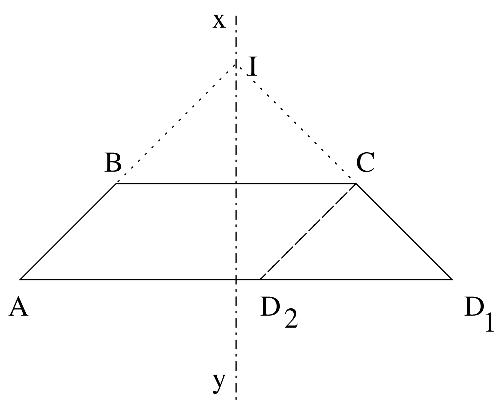

FIGURE 45

together with A, B, and C forms a parallelogram. These oblique lines certainly have endpoints on AD, since that line is parallel to BC. It follows that D must be either D_1 or D_2, since there are only two oblique lines equal to AB from C to AD.

The reasoning above may seem to be incorrect if D_2 coincides with D_1; that is, when CD_1 is parallel to AB. This requires, as we have just seen, that AB also be parallel to xy, and therefore perpendicular to AD, and indeed in this case there is only one oblique line from C with length equal to AB (the perpendicular).

REMARK II. In the first part of the preceding converse it is essential that the quadrilateral be proper (**21**, Remark): it is only if the triangles ABC, ADC of Fig. 44 are on different sides of the common side AC that the angles $\widehat{A_1}$, $\widehat{C_1}$ are alternate interior.

It is easy to construct an improper quadrilateral (called a *anti-parallelogram*) whose opposite sides are equal. It suffices, in parallelogram $ABCD$ (Fig. 45b), to replace the point D by point E symmetric to it with respect to the diagonal AC.

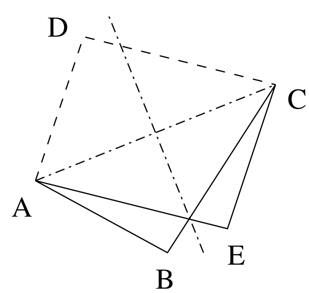

FIGURE 45b

We can also obtain an anti-parallelogram $ABCE_1$ by taking E_1 to be symmetric to B with respect to the perpendicular bisector of AC, so that ABE_1C is an isosceles trapezoid. But this point E_1 is none other than E, because (**24**, case 3°) there is, on any one side of AC, only one point E which is both at a distance $AE = BC$ from A and also at a distance $CE = AB$ from C. Every improper quadrilateral with equal opposite sides is thus formed by the non-parallel sides and the diagonals of an isosceles trapezoid.

Remark III. Two quadrilaterals with four pairs of equal sides need not be congruent. In other words, we can deform a quadrilateral $ABCD$ (proper or not) without changing the lengths of its sides.

Indeed, let $AB = a$, $BC = b$, $CD = c$, $DA = d$ be the constant lengths of the four sides. With sides a, b and an arbitrary included angle B, we can construct a triangle ABC, whose construction will determine the length of AC (a diagonal of the quadrilateral). To each value of this diagonal (under the conditions for existence given in **86**, Book II) there corresponds a triangle ACD with this base AC and with the other two sides CD, DA having length c, d. The angle $\widehat{B}$ can therefore be arbitrary (at least within certain limits).

A quadrilateral which can be deformed under these conditions is called an *articulated quadrilateral.* This notion is important in practical applications of geometry.

According to the preceding results, a parallelogram remains a parallelogram when it is articulated and, likewise, an anti-parallelogram remains an anti-parallelogram under these conditions.[1]

47. Theorem. *In a parallelogram, the two diagonals divide each other into equal parts.*

In parallelogram $ABCD$ (Fig. 46), we draw diagonals AC, BD, which intersect at O. Triangles ABO, CDO are congruent because they have equal angles, and an equal side $AB = CD$ (by the preceding theorem). Therefore $AO = CO$, $BO = DO$. QED

Converse. *A quadrilateral is a parallelogram if its diagonals divide each other into equal parts.*

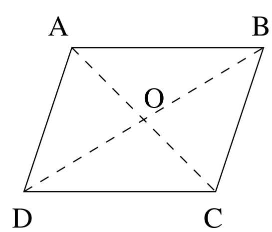

Figure 46

Indeed, assume (Fig. 46) that $AO = CO$, $BO = DO$. Triangles ABO, CDO are again congruent because they have an equal angle (the angles at O are vertical) included between pairs of equal sides. Therefore their angles at A and C will be equal, so that AB is parallel to CD. The congruence of triangles ADO, BCO also shows that AD is parallel to BC.

[1]This reasoning fails only if a parallelogram is transformed into an anti-parallelogram, or inversely (since a parallelogram and an anti-parallelogram are the only quadrilaterals with opposite sides equal). If the deformation occurs continuously, this requires that the quadrilateral first flatten into a line, so that one pair of adjacent sides ends up as extensions of each other, as does the other pair.

Remark. We have proved the converses in **46** and **47** by retracing the original reasoning in the opposite direction, as explained in **32**, Remark II.

48. A quadrilateral whose angles are all equal, and are therefore are all right angles, is called a *rectangle*. A rectangle is a parallelogram, since its opposite angles are equal.

A quadrilateral whose sides are all equal is called a *rhombus*. A rhombus is a parallelogram since it has equal opposite sides.

Thus, in a rectangle, as in a rhombus, the diagonals intersect at their common midpoint.

Theorem. *The diagonals of a rectangle are equal.*

In rectangle $ABCD$ (Fig. 47), the diagonals are equal because triangles ACD, BCD are congruent: they have side DC in common, $\widehat{ADC} = \widehat{DCB}$ since they are both right angles, and $AD = BC$ since they are opposite sides of a parallelogram.

Corollary. *In a right triangle, the median from the vertex of the right angle equals half the hypotenuse.*

This is true because if we draw parallels to the sides of the right angle through the endpoints of the hypotenuse, we form a rectangle, in which the median in question is half the diagonal.

Converse. *A parallelogram with equal diagonals is a rectangle.*

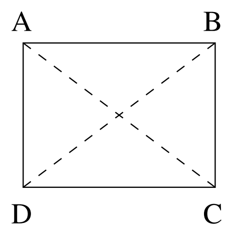

Figure 47

Suppose (Fig. 47) that $AD = BC$ in parallelogram $ABCD$. We know that $AD = BC$: consequently, triangles ADC, BCD are congruent, since their three sides are equal in pairs. Angles $\widehat{ADC}$, $\widehat{BCD}$ are therefore equal and, since they are supplementary, they must be right angles, which shows that the parallelogram is a rectangle.

Corollary. *A triangle in which a median is one half the corresponding side is a right triangle.*

Corollary. *In a rhombus, the diagonals are perpendicular, and bisect the vertex angles.*

If $ABCD$ (Fig. 48) is a rhombus, then triangle ABD is isosceles. Diagonal AC, being a median of this triangle, is also an altitude and an angle bisector.

Converse. *A parallelogram with perpendicular diagonals is a rhombus.*

Indeed, each vertex is equidistant from the adjacent vertices, since it lies on the perpendicular bisector of the diagonal joining these vertices.

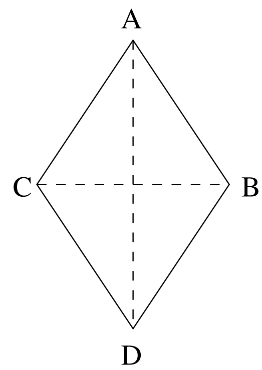

FIGURE 48

49. A *square* is a quadrilateral in which all sides are equal, and all the angles right angles.

Thus a square is both a rhombus and a rectangle, so that the diagonals are equal, perpendicular, and intersect each other at their common midpoint.

Conversely, any quadrilateral whose diagonals are equal, perpendicular, and bisect each other is a square.

Two squares with the same side are congruent.

50. Translations.

LEMMA. *Two figures F, F' are congruent, with the same sense of rotation, if their points correspond in such a way that, if we take points A, B, C from one figure, and the corresponding points A', B', C' from the other, then the triangles thus formed are always congruent, with the same sense of rotation, no matter which point we take for C.*

Indeed, take two points A, B in figure F, and the homologous[2] points A', B'. Segment AB is then clearly equal to $A'B'$. Let us move the second figure onto the first in such a way that these two equal segments coincide. We claim that then the two figures coincide completely. Indeed, let C be a third point of the first figure, and let C' be its homologous point. Since the two triangles ABC, $A'B'C'$ are congruent, angle $\widehat{B'A'C'}$ is equal to $\widehat{BAC}$, and has the same sense of rotation. Therefore, when $A'B'$ is made to coincide with AB, the line $A'C'$ will assume the direction of AC. Since $A'C' = AC$ as well, we conclude that C' coincides with C. This argument applies to all the points of the figure, so the two figures must coincide completely.

REMARKS. I. We have just provided a sufficient condition for two figures to be congruent; this conditions is also clearly necessary.

From the preceding reasoning it also follows that:

II. In order to superimpose two equal figures with the same orientation, it suffices to superimpose two points of one of the figures onto their homologous points.

[2]This is the name given to pairs of corresponding points in the two figures.

51. THEOREM. *If, starting at each point of a figure, we draw equal parallel segments in the same direction, then their endpoints will form a figure congruent to the first.*

First consider two points A, B of the first figure, and the corresponding points A', B' of the second. Since the segments AA' and BB' are parallel and equal, $ABA'B'$ is a parallelogram. Therefore $A'B'$ is equal and parallel to AB, with the same orientation. Thus segments joining homologous pairs of points are equal, parallel, and have the same direction.

It follows that any three points of the first figure correspond to three points forming a congruent triangle, and since the angles of these triangles have their sides parallel and in the same sense, their sense of rotation is the same. The figures are therefore congruent.

The operation by which we pass from the first figure to the second is called a *translation.* We note that a translation is determined if we are given the length, direction, and sense of a segment, such as AA', joining a point to its homologue. We can therefore designate a translation by the letters of such a segment: for example, we speak of the translation AA'.

COROLLARIES. I. *If, through each point on a line, we draw equal parallel segments in the same sense, the locus of their endpoints is a line parallel to the first.*

In particular, *the locus of points on the same side of a line, and a at a given distance from the line, is a parallel line.*

II. *Two parallel lines are everywhere equidistant.*

We can therefore speak of the *distance* between two parallel lines.

III. *The locus of points equidistant from two parallel lines is a third line, parallel to the first two.*

Exercises

Parallelograms.

Exercise 26. The angle bisectors of a parallelogram form a rectangle. The bisectors of the exterior angles also form a rectangle.

Exercise 27. Any line passing through the intersection of the diagonals of a parallelogram is divided by this point, and by two opposite sides, into two equal segments.

For this reason, the point of intersection of the diagonals of a parallelogram is called the *center* of this polygon.

Exercise 28. Two parallelograms, one of which is *inscribed* in the other (that is, the vertices of the second are on the sides of the first) must have the same center.

Exercise 29. An angle of a triangle is acute, right, or obtuse, according as whether its opposite side is less than, equal to, or greater than double the corresponding median.

Exercise 30. If, in a right triangle, one of the acute angles is double the other, then one of the sides of the right angle is half the hypotenuse.

Translations.

Exercise 31. Find the locus of the points such that the sum or difference of its distances to two given lines is equal to a given length.

Exercise 32. Given two parallel lines, and two points A, B outside these two parallels, and on different sides, what is the shortest broken line joining the two points, so that the portion contained between the two parallels has a given direction?

CHAPTER VII

Congruent Lines in a Triangle

52. Theorem. *In any triangle, the perpendicular bisectors of the three sides are concurrent.*

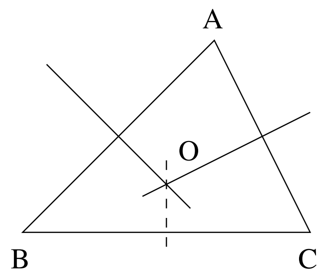

Figure 49

Suppose the triangle is ABC (Fig. 49). The perpendicular bisectors of sides AB, AC are not parallel (otherwise lines AB, AC would coincide), so they intersect at some point O. We must show that point O is also on the perpendicular bisector of BC.

Because point O, is on the perpendicular bisector of AB, it is equidistant from A and B; likewise, because it is on the perpendicular bisector of AC, it is equidistant from A and C. It is therefore equidistant from B and C and thus it is on the perpendicular bisector of BC.

53. Theorem. *In any triangle, the three altitudes are concurrent.*

Suppose the triangle is ABC (Fig. 50). We draw a parallel to BC through A, a parallel to AC through B, and a parallel to AB through C. This forms a new triangle $A'B'C'$. We will show that the altitudes of ABC are the perpendicular bisectors of the sides of the new triangle, from which it follows that they are concurrent.

Parallelogram $ABCB'$ gives us $BC = AB'$, and parallelogram $ACBC'$ gives $BC = AC'$, so that A is indeed the midpoint of $B'C'$. Altitude AD of ABC thus passes through the midpoint of $B'C'$, and is perpendicular to it because it is perpendicular to the parallel line BC.

Since this reasoning can be repeated for the other altitudes, the proof is complete.

54. Theorem. *In any triangle:*

1°. *the three angle bisectors are concurrent;*

2°. *the bisector of an angle, and the bisectors of the two non-adjacent exterior angles, are concurrent.*

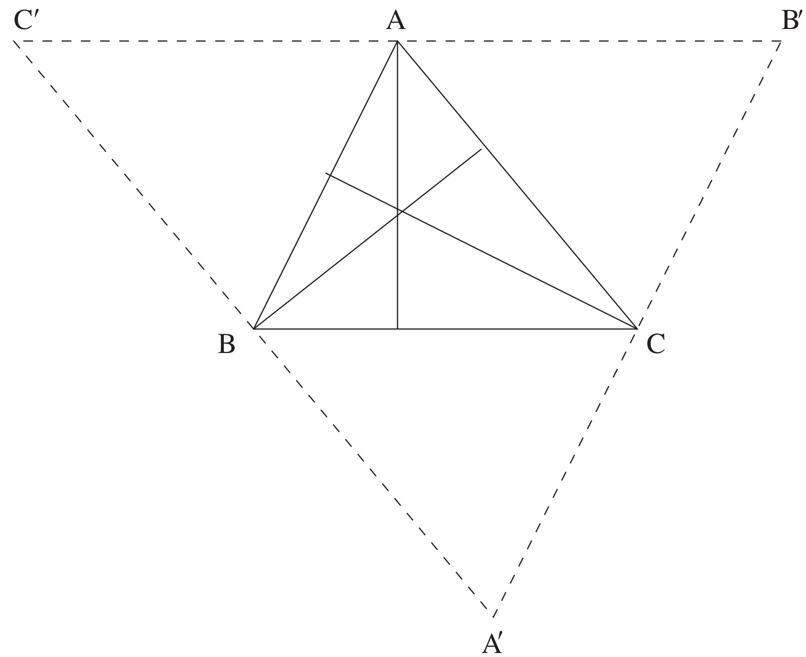

FIGURE 50

1°. In triangle ABC (Fig.51), we draw the bisectors of angles $\widehat{B}$ and $\widehat{C}$; they intersect at a point O inside the triangle. This point, being on the bisector of angle $\widehat{B}$, is equidistant from the sides AB and BC. Likewise, being on the bisector of angle $\widehat{C}$, the point O is equidistant from AC and BC. It is therefore equidistant from AB and AC and, being interior to angle $\widehat{A}$, is on the bisector of this angle.

2°. Since the exterior angles $\widehat{CBx}$, $\widehat{BCy}$ have a sum less than four right angles, halves of each have a sum less than two right angles. The bisectors of these angles will therefore meet (**41**, Corollary) at a point O' inside angle $\widehat{A}$. This point O', like point O, will be equidistant from the three sides of the triangle. Therefore it will belong to the bisector of $\widehat{A}$.

55. THEOREM. *The segment joining the midpoints of two sides of a triangle is parallel to the third side, and equal to half of it.*

In triangle ABC (Fig. 52), let D be the midpoint of AB and let E be the midpoint of AC. We extend line DE past E by its own length, to a point F. Quadrilateral $ADCF$ will be a parallelogram (**47**), and therefore CF will be equal and parallel to DA, or, equivalently, to BD. Thus quadrilateral $BDCF$ is also a parallelogram. Therefore:

1°. DE is parallel to BC;

2°. DE, which is equal to half of DF, is also half of BC.

56. THEOREM. *The three medians of a triangle are concurrent at a point situated on each of them one third of its length from the corresponding side.*

First, let BE, CF be two medians of triangle ABC (Fig. 53). We claim that their intersection G lies at one third the length of each of them. To see this, let M

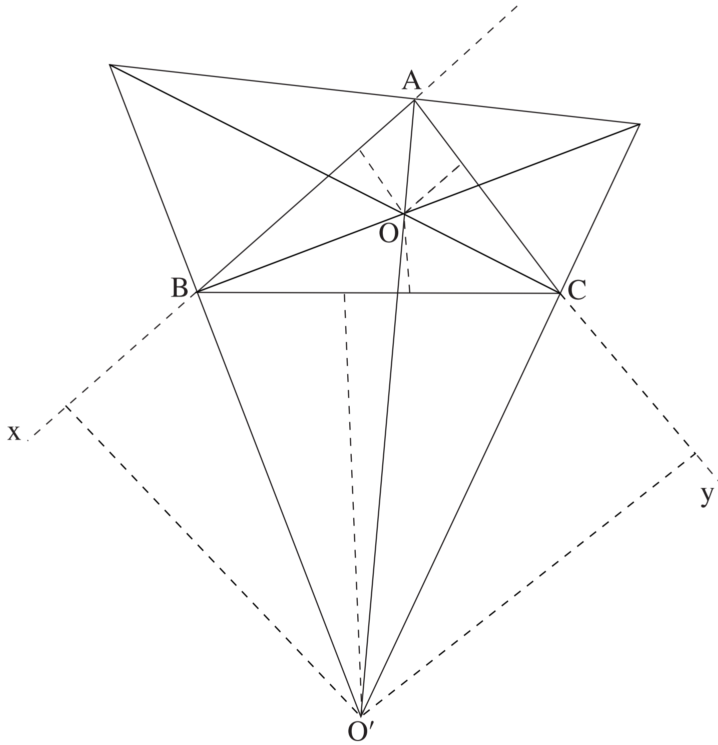

FIGURE 51

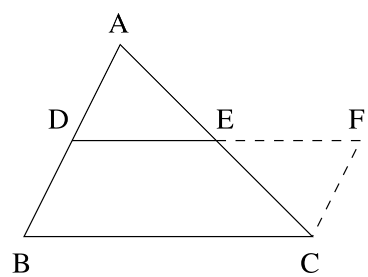

FIGURE 52

and N be the midpoints of BG and CG. Segment MN, which joins the midpoints of two sides of triangle BCG, is parallel to BC and equal to half of it. But EF is parallel to BC and equal to half of it. This means that $EFMN$ is a parallelogram, whose diagonals divide each other in half. Therefore $EG = GM = MB$ and $FG = GN = NC$.

Thus median BE passes through the point situated at one third the length of CF. But the same reasoning can also be applied to show that the median AD passes through the same point. The thereom is proved.

REMARK. The point where the medians meet is also called the *center of mass* of the triangle. The reason for this name is given in the study of mechanics.

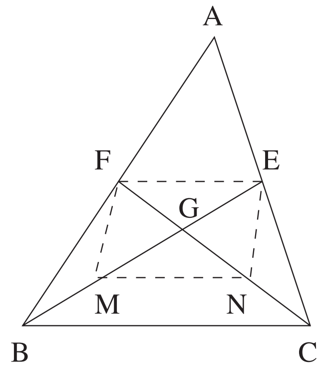

FIGURE 53

Exercises

Exercise 33. Join a given point to the intersection of two lines, which intersect outside the limits of the diagram (**53**).

Exercise 34. In a trapezoid, the midpoints of the non-parallel sides and the midpoints of the two diagonals are on the same line, parallel to the bases. The distance between the midpoints of the non-parallel sides equals half the sum of the bases; the distance between the midpoints of the diagonals is equal to half their difference.

Exercise 35. If, from two points A, B and the midpoint C of AB, we drop perpendiculars onto an arbitrary line, the perpendicular from C is equal to half the sum of the other two perpendiculars, or to half their difference, according as whether these two perpendiculars have the same or opposite sense.

Exercise 36. The midpoints of the sides of any quadrilateral are the vertices of a parallelogram. The sides of this parallelogram are parallel to the diagonals of the given quadrilateral, and equal to halves of these diagonals. The center of the parallelogram is also the midpoint of the segment joining the midpoints of the diagonals of the given quadrilateral.

Exercise 37. Prove that the medians of a triangle ABC are concurrent by extending the median CF (Fig. 53) beyond F by a length equal to FG.

Exercise 38. Given three lines passing through the same point O (all three distinct), and a point A on one of them, show that there exists:

1°. A triangle with a vertex at A and having the three lines as its altitudes (one exception);

2°. A triangle with a vertex at A and having the three lines as its medians;

3°. A triangle with a vertex at A and having the three lines as bisectors of its interior or exterior angles (one exception);

4°. A triangle with a midpoint of one of its sides at point A and having the three lines as perpendicular bisectors of the sides (reduce this to 1°).

Problems for Book 1

Exercise 39. In a triangle, the larger side corresponds to the smaller median[1]. (Consider the angle made by the third median with the third side.) A triangle with two equal median is isosceles.

Exercise 40. Let us assume that a billiard ball which strikes a flat wall will bounce off in such a way that the two lines of the path followed by the ball (before and after the collision) make equal angles with the wall. Consider n lines $D_1, D_2, \ldots, D_n$ in the plane, and points A, B on the same side of all of these lines. In what direction should a billiard ball be shot from A in order that it arrive at B after having bounced off each of the given lines successively? Show that the path followed by the ball in this case is the shortest broken line going from A to B and having successive vertices on the given lines.[2]

Special Case. The given lines are the four sides of rectangle, taken in their natural order; the point B coincides with A and is inside the rectangle. Show that, in this case, the path traveled by the ball is equal to the sum of the diagonals of the rectangle.

Exercise 41. The diagonals of the two rectangles of exercise 26 are situated on the same two lines, parallel to the sides of the given parallelogram (analogous to **54**). One of these diagonals is half the difference, and the other half the sum, of the sides of the parallelogram.

Exercise 42. In an isosceles triangle, the sum of the distances from a point on the base to the other sides is constant.— What happens if the point is taken on the extension of the base?

In an equilateral triangle, the sum of the distances from a point inside the triangle to the three sides is constant. What happens when the point is outside the triangle?

Exercise 43. In triangle ABC, we draw a perpendicular through the midpoint D of BC to the bisector of angle A. This line cuts off segments on the sides AB, AC equal to, respectively, $\frac{1}{2}(AB + AC)$ and $\frac{1}{2}(AB - AC)$.

Exercise 44. Let $ABCD$, $DEFG$ be two squares placed side by side, so that sides DC, DE have the same direction, and sides AD, DG are extensions of each other. On AD and on the extension of DC, we take two segments AH, CK equal to DG. Show that quadrilateral $HBKF$ is also a square.

Exercise 45. On the sides AB, AC of a triangle, and outside the triangle, we construct squares $ABDE$, $ACGF$, with D and F being the vertices opposite A. Show that:

1°. EG is perpendicular to the median from A, and equal to twice this median;

2°. The fourth vertex I of the parallelogram with vertices EAG (with E and G opposite vertices) lies on the altitude from A in the original triangle;

[1] For similar statements concerning the altitudes of a triangle, see Exercises 19, 20, and for angle bisectors see exercises 362, 362b at the end of this volume.

[2] In the case where there is only one line, the problem reduces to the subject of Exercises 13–14. Once these exercises are solved, one tries to find a way to use the solution for the case of one line to treat the case for two lines; then to extend it to three lines, and so on.

3°. CD, BF are equal to and perpendicular to BI, CI respectively, and their intersection point is also on the altitude from A.

Exercise 46. We are given a right angle $\widehat{AOB}$, and two perpendicular lines through a point P, the first intersecting the sides of the angle in A, B, and the second intersecting the same sides in C, D. Show that the perpendiculars from the points D, O, C to the line OP intercept on AB segments equal to AP, PB, respectively, but having the opposite sense.