Preface

One day, in 1918, G.H. Hardy, the great English mathematician, took what he thought was an ordinary cab ride to go visit his young protégé at the hospital, the Indian mathematician S. Ramanujan. To break the ice, Hardy mentioned that the number 1729 on his taxicab was a rather dull number. Ramanujan immediately replied that, on the contrary, it was a very fascinating number since it was the smallest positive integer which could be written as the sum of two cubes in two distinct ways: $1729 = 12^3 + 1^3 = 10^3 + 9^3$. This anecdote certainly shows the genius of Ramanujan, but it also stirs our imagination. In some sense, it challenges us to find the remarkable characteristics of other numbers.

This is precisely the task we undertake in this project. The reader will find here "famous" numbers such as 1729, Mersenne prime numbers (those prime numbers of the form $2^p - 1$, where p is itself a prime number) and perfect numbers (those numbers equal to the sum of their proper divisors); also "less famous" numbers, but no less fascinating, such as the following ones:

- 37, the median value of the second prime factor of an integer; thus, the probability that the second prime factor of an integer chosen at random is smaller than 37 is approximately $\frac{1}{2}$;
- 277, the smallest prime number p which allows the sum

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \ldots + \frac{1}{p}$$

(where the sum is running over all the prime numbers $\leq p$) to exceed 2;

- 378, the smallest prime number which is not a cube, but which can be written as the sum of the cubes of its prime factors: indeed, $378 = 2 \cdot 3^3 \cdot 7 = 2^3 + 3^3 + 7^3$;
- 480, possibly the largest number n such that $n(n+1)\ldots(n+5)$ has exactly the same distinct prime factors as $(n+1)(n+2)\ldots(n+6)$; indeed,

$$\begin{aligned} 480 \cdot 481 \cdot \ldots \cdot 485 &= 2^8 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot 23 \cdot 37 \cdot 97 \cdot 241, \\ 481 \cdot 482 \cdot \ldots \cdot 486 &= 2^4 \cdot 3^6 \cdot 5 \cdot 7 \cdot 11^2 \cdot 13 \cdot 23 \cdot 37 \cdot 97 \cdot 241; \end{aligned}$$

- 736, the only three digit number abc such that $abc = a+b^c$; indeed, $736 = 7+3^6$;
- 1 782, possibly the only integer $n > 1$ for which $\prod_{p|n} p = \sqrt{\sum_{d|n} d}$;
- 548 834, the only number > 1 which can be written as the sum of the sixth powers of its digits: indeed, $548\,834 = 5^6 + 4^6 + 8^6 + 8^6 + 3^6 + 4^6$;
- 11 859 210, the smallest number n for which $P(n)^4|n$ and $P(n+1)^4|(n+1)$, where $P(n)$ stands for the largest prime factor of n (here $P(n) = 11$ and $P(n+1) = 19$); the second smallest known number n satisfying this property is $n = 632\,127\,050\,601\,113\,666\,430$ (here $P(n) = 2131$ and $P(n+1) = 3691$);
- 89 460 294, the smallest number n (and the only one known) for which $\beta(n) = \beta(n+1) = \beta(n+2)$, where $\beta(n)$ stands for the sum of the distinct prime factors of n;
- 305 635 357, the smallest composite number n for which $\sigma(n+4) = \sigma(n) + 4$, where $\sigma(n)$ stands for the sum of the divisors of n;
- 612 220 032, the smallest number $n > 1$ whose sum of digits is equal to $\sqrt[7]{n}$;
- 3 262 811 042, possibly the only number which can be written as the sum of the fourth powers of two prime numbers in two distinct ways: $3\,262\,811\,042 = 7^4 + 239^4 = 157^4 + 227^4$;
- 3 569 485 920, the number n at which the expression $\dfrac{\Omega(n)^{\omega(n)}}{n}$ reaches its maximum value, namely 2.97088..., where $\omega(n)$ stands for the number of distinct prime factors of n and $\Omega(n)$ stands for the number of prime factors of n counting their multiplicity.

Various numbers also raise interesting issues. For instance, does there exist a number which is not the square of a prime number but which can be written as the sum of the squares of its prime factors? Given an arbitrary integer $k \geq 2$, does there exist a number n such that $P(n)^k|n$ and $P(n+1)^k|(n+1)$? For each integer $k \geq 2$ which is not a multiple of 3, can one always find a prime number whose sum of digits is equal to k? These are some of the numerous open problems stated in this book, each of them standing for an enigma that will certainly feed the curiosity of the reader. Actually my hope for this book is to encourage many to explore more thoroughly some of the questions raised all along this book.

There are currently several books whose main purpose is to exhibit interesting properties of numbers. This book is along the lines of these works but offers more features. For instance, one will find – mainly in the footnotes – short proofs of key results as well as statements of many new open problems.

Finally, I would like to acknowledge all those who contributed to this manuscript. With their precious input, suggestions and ideas, this project was expansive but enjoyable. Thanks to Jean-Lou De Carufel, Charles Cassidy, Zita De Koninck, Éric Doddridge, Nicolas Doyon, Éric Drolet, David Grégoire, Bernard Hodgson, Imre Kátai, Patrick Letendre, Claude Levesque, Florian Luca, Michael Murphy, Erik Pronovost and Jérôme Soucy.

This edition is a translation of my French book *Ces nombres qui nous fascinent* published by Ellipses in 2008.

Anyone enjoying this book is welcome to send me suggestions and ideas which could improve and enlighten this project.

Jean-Marie De Koninck
Département de mathématiques et de statistique
Université Laval
Québec G1V 0A6
CANADA

jmdk@mat.ulaval.ca