Those Fascinating Numbers

1

- the only number which divides all the others.

2

- the only even prime number.

3

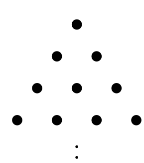
- the prime number which appears the most often as the second prime factor of an integer, and actually with a frequency of $\frac{1}{6}$ (see the number 199 for the list of those prime numbers which appear the most often as the k^{th} prime factor of an integer, for any fixed $k \geq 1$).

- the smallest Mersenne prime ($3 = 2^2 - 1$): a prime number is called a *Mersenne prime* if it is of the form $2^p - 1$, where p is prime (see the number 131 071 for the list of all Mersenne primes known as of May 2009);

- the prime number which appears the most often as the second largest prime factor of an integer, that is approximately $(1 + \log 2 + \frac{3}{2} \log 3)x/\log x$ times amongst the positive integers $n \leq x$ (see J.M. De Koninck [44]);

- the smallest triangular number > 1: a number n is said to be *triangular* if there exists a number k such that

$$n = 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$$



4

- the smallest number r which has the property that each number can be written in the form $x_1^2 + x_2^2 + \ldots + x_r^2$, where the x_i's are non negative integers; the problem consisting in determining if, for a given integer $k \geq 2$, there exists a number r (depending only on k) such that equation

$$(*) \qquad n = x_1^k + x_2^k + \ldots + x_r^k$$

has solutions for each number n, is due to the English mathematician E. Waring who, in 1770, stated without proof that "each number is the sum of 4 squares, of 9 cubes, of 19 fourth powers, and so on"; if we denote by $g(k)$ the smallest number r such that equation $(*)$ has solutions for each number n, Lagrange proved in 1770 that $g(2) = 4$, Wieferich and Kempner proved around 1910 that $g(3) = 9$, while R. Balasubramanian, J.M. Deshouillers & F. Dress [12] proved in 1986 that $g(4) = 19$; it is conjectured that $g(k) = 2^k + [(3/2)^k] - 2$ (where $[x]$ stands for the largest integer $\leq x$) for each integer $k \geq 2$; see L.E. Dickson [65])[1]; hence by using this formula, we find that the values of $g(k)$, for $k = 1, 2, \ldots, 20$, are respectively 1, 4, 9, 19, 37, 73, 143, 279, 548, 1079, 2132, 4223, 8384, 16673, 33203, 66190, 132055, 263619, 526502, 1051899 (see the book of Eric Weisstein [201], p. 1917).

5

- the smallest Wilson prime: a prime number p is called a *Wilson prime* if it satisfies the congruence $(p-1)! \equiv -1 \pmod{p^2}$: the only known Wilson primes are 5, 13 and 563; K. Dilcher & C. Pomerance [68] have shown that there are no other Wilson primes up to $5 \cdot 10^8$.

6

- the smallest perfect number: a number n is said to be *perfect* if it is equal to the sum of its proper divisors, that is if $\sigma(n) = 2n$; the sequence of perfect numbers starts as follows: 6, 28, 496, 8 128, 33 550 336, ...; a number n is said to be *k-perfect* if $\sigma(n) = kn$: if we let n_k stand for the smallest k-perfect number, then $n_2 = 6$, $n_3 = 120$, $n_4 = 30\,240$, $n_5 = 14\,182\,439\,040$ and $n_6 = 154\,345\,556\,085\,770\,649\,600$;

- the smallest unitary perfect number: a number n is said to be a *unitary perfect number* if $\displaystyle\sum_{\substack{d \mid n \\ (d, n/d)=1}} d = 2n$, where $(d, n/d)$ stands for the greatest common divisor of d and n/d; only five unitary perfect numbers are known, namely 6, 60, 90, 87 360 and $146\,361\,946\,186\,458\,562\,560\,000 = 2^{18} \cdot 3 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313$: this last number was discovered by C.R. Wall [198] (see also R.K. Guy [101], B3);

[1] In 1936, S. Pillai [161] proved that if one writes $3^k = q2^k + r$ with $0 < r < 2^k$, then $g(k) = 2^k + [(3/2)^k] - 2$ provided $r + q \leq 2^k$.

- the only triangular number > 1 whose square is also a triangular number (W. Ljunggren, 1946): here $6^2 = 36 = 1 + 2 + 3 + \ldots + 8$.

7

- one of the two prime numbers (the other one is 5) which appears most often as the third prime factor of an integer (1 time in 30);
- the second Mersenne prime: $7 = 2^3 - 1$.

8

- the third number n such that $\tau(n) = \phi(n)$: the only numbers satisfying this equation are 1, 3, 8, 10, 18, 24 and 30;
- the number of twin prime pairs < 100 (see the number 1 224).

9

- the only square which follows[2] a power of 2: $2^3 + 1 = 3^2$;
- the only perfect square which cannot be written as the sum of four squares (Sierpinski [185], p. 405);
- the smallest number r which has the property that each number can be written as $x_1^3 + x_2^3 + \ldots + x_r^3$, where the x_i's are non negative integers (see the number 4).

10

- one of the five numbers (the others are 1, 120, 1 540 and 7 140) which are both triangular and tetrahedral (see E.T. Avanesov [8]): a number n is said to be *tetrahedral* if it can be written as $n = \frac{1}{6}m(m+1)(m+2)$ for some number m: it corresponds to the number of spheres with same radius which can be piled up in a tetrahedron;
- the fourth number n such that $\tau(n) = \phi(n)$ (see the number 8).

[2] Much more is known. Indeed, according to the Catalan Conjecture (first stated by Catalan [31] in 1844), *the only consecutive numbers in the sequence of powers* $1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, \ldots$ *are 8 and 9;* this conjecture was recently proved by Preda Mihailescu [135].

11

- the smallest prime number p such that $3^{p-1} \equiv 1 \pmod{p^2}$: the only other prime number $p < 2^{32}$ satisfying this congruence is $p = 1\,006\,003$ (see Ribenboim [169], p. 347)[3];
- the smallest number n which allows the sum $\sum_{i \le n} \frac{1}{i}$ to exceed 3 (see the number 83).

12

- the smallest pseudo-perfect number: we say that a number is *pseudo-perfect* if it can be written as the sum of some of its proper divisors: here $12 = 6 + 4 + 2$; in 1976, Erdős proved that the set of pseudo-perfect numbers is of positive density (see R.K. Guy [101], B2);
- the smallest number m for which equation $\sigma(x) = m$ has exactly two solutions, namely 6 and 11;
- the only number $n > 1$ such that $\sigma(\gamma(n)) = n$;
- the smallest sublime number: we say that a number n is *sublime* if $\tau(n)$ and $\sigma(n)$ are both perfect numbers: here $\tau(12) = 6$ and $\sigma(12) = 28$; this concept was introduced by Kevin Ford; the only other known sublime number is $2^{126}(2^{61} - 1)(2^{31} - 1)(2^{19} - 1)(2^7 - 1)(2^5 - 1)(2^3 - 1)$.

13

- the second Wilson prime (see the number 5);
- the prime number which appears the most often as the fourth prime factor of an integer, namely 31 times in 5005 (see the number 199);
- the smallest prime number p such that $23^{p-1} \equiv 1 \pmod{p^2}$: the only prime numbers $p < 2^{32}$ satisfying this congruence are 13, 2 481 757 and 13 703 077 (see Ribenboim [169], p. 347);
- the third horse number: we say that n is a *horse number* if it represents the number of possible results accounting for ties, in a race in which k horses participate; thus, if H_k is the k^{th} horse number, one can prove[4] that

$$H_k = \sum_{i=1}^{k} i^k \left(\sum_{j=0}^{k-i} (-1)^j \binom{j+i}{j} \right);$$

  the first 20 terms of the sequence $(H_k)_{k \ge 1}$ are 1, 3, 13, 75, 541, 4683, 47293, 545835, 7087261, 102247563, 1622632573, 28091567595, 526858348381, 10641342970443, 230283190977853, 5315654681981355, 130370767029135901, 3385534663256845323, 92801587319328411133 and 2677687796244384203115.

[3] As is the case for the Wieferich primes (see the number 1 093), it is not known if this sequence of numbers is infinite.

[4] A formula established by Charles Cassidy (Université Laval).

14

- the smallest solution[5] of $\sigma(n) = \sigma(n+1)$; the sequence of numbers satisfying this equation begins as follows: 14, 206, 957, 1334, 1364, 1634, 2685, 2974, 4364, 14841, 18873, 19358, 20145, 24957, 33998, 36566, 42818, 56564, 64665, 74918, 79826, 79833, 84134, 92685, ...;
- the fourth Catalan number: Catalan numbers[6] are the numbers of the form $\frac{1}{n+1}\binom{2n}{n}$.

15

- the third smallest solution of $\phi(n) = \phi(n+1)$; the sequence of numbers satisfying this equation begins as follows: 1, 3, 15, 104, 164, 194, 255, 495, 584, 975, 2204, 2625, 2834, 3255, 3705, 5186, 5187, 10604, 11715, 13365, 18315, 22935, 25545, 32864, 38804, 39524, 46215, 48704, 49215, 49335, 56864, 57584, 57645, 64004, 65535, 73124, ...: R. Baillie [10] found 391 solutions $n < 2 \cdot 10^8$;[7]
- one of the three numbers n such that the polynomial $x^5 - x \pm n$ can be factored: the other two are $n = 22\,440$ and $n = 2\,759\,640$: here we have $x^5 - x \pm 15 = (x^2 \pm x + 3)(x^3 \mp x^2 - 2x \pm 5)$; see the number 22 440;
- the value of the sum of the elements of a diagonal, a row or a column of a 3×3 magic square: for a $k \times k$ magic square with $k \geq 3$, the common value is $k(k^2+1)/2$, which gives place to the sequence whose first terms are 15, 34, 65, 111, 175, 260, 369, 505, 671, 870, 1105, 1379, 1695, ... (see Sierpinski [185], p. 434).

16

- the only number n for which there exist two distinct integers a and b such that $n = a^b = b^a$: here $a = 2$, $b = 4$;
- the smallest perfect square for which there exists another perfect square with the same sum of divisors: $\sigma(16) = \sigma(25) = 31$.

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[5] This sequence of numbers is probably infinite, but no one has yet proved it.

[6] Catalan numbers appear when one wants to find in how many ways it is possible to partition a convex polygon in triangles by drawing some of its diagonals.

[7] P. Erdős, C. Pomerance & A. Sárközy [79] provide a heuristic argument which suggests that, for each fixed $\varepsilon > 0$, equation $\phi(n) = \phi(n+1)$ has at least $x^{1-\varepsilon}$ solutions $n \leq x$. However, A. Schinzel [180] believes that it may be possible that equation $\phi(n) = \phi(n+1)$ has only a finite number of solutions, but he conjectures that for each even integer $k \geq 2$, equation $\phi(n) = \phi(n+k)$ has infinitely many solutions. Let us add that equation $\phi(n) = \phi(n+k)$ has very few solutions when k is odd and divisible by 3; thus by letting E_k be the set of solutions $n < 10^8$ of $\phi(n) = \phi(n+k)$, we have $E_3 = \{3,5\}$, $E_9 = \{9,15\}$, $E_{15} = \{13,15,17,21\}$, $E_{21} = \{21,35\}$ and $E_{27} = \{27,45,55\}$, while the cardinality of each of the other sets E_k, $1 \leq k \leq 32$, is at least 12.

17

- the third Fermat prime ($17 = 2^{2^2}+1$), the first two being 3 and 5: a number of the form $2^{2^k}+1$, where k is a non negative integer, is called a *Fermat number* and is often denoted by F_k (see the number 70 525 124 609); if such a number is prime, we say that it is a *Fermat prime*;
- the only prime number which is the sum of four consecutive prime numbers: $17 = 2+3+5+7$;
- the exponent of the sixth Mersenne prime ($131\,071 = 2^{17}-1$) (Cataldi, 1588);
- the smallest Stern number (see the number 137).

18

- the largest known number x for which there exist numbers $n \geq 3$, y and $q \geq 2$ such that $(x^n-1)/(x-1) = y^q$; the only known solutions of this last equation are given by

$$\frac{3^5-1}{3-1} = 11^2, \qquad \frac{7^4-1}{7-1} = 20^2, \qquad \frac{18^3-1}{18-1} = 7^3$$

  (see Y. Bugeaud, M. Mignotte & Y. Roy [26]);
- the fifth number n such that $\tau(n) = \phi(n)$ (see the number 8).

19

- the smallest number r which has the property that each number can be written as $x_1^4+x_2^4+\ldots+x_r^4$, where the x_i's are non negative integers (see the number 4);
- one of the nine known numbers k such that $\underbrace{11\ldots1}_{k}$ is prime: the others[8] are 2, 23, 317, 1 031, 49 081, 86 453, 109 297 and 270 343;
- the largest known prime p_k such that $\nu(p_k) := \prod_{p \leq p_k} \frac{p+1}{p-1}$ is an integer: here, $\nu(p_8) = \nu(19) = 21$;
- the exponent of the seventh Mersenne prime ($524\,287 = 2^{19}-1$) (Cataldi, 1588).

20

- the smallest solution of $\sigma(n) = \sigma(n+6)$; the sequence of numbers satisfying this equation begins as follows: 20, 155, 182, 184, 203, 264, 621, 650, 702, 852, 893, 944, 1343, 1357, 2024, 2544, 2990, 4130, 4183, 4450, 5428, 5835, 6149, 6313, 6572, 8177, 8695, ...

[8] Such a number k must be a prime, for if it was not, then we would have $k = ab$ with $1 < a \leq b < k$, in which case $\frac{10^{ab}-1}{9} = \frac{10^{ab}-1}{10^b-1} \cdot \frac{10^b-1}{9}$, the product of two numbers > 1.

21

- the smallest integer > 1 whose sum of divisors is a fifth power: here $\sigma(21) = 2^5$;
- the smallest 2-hyperperfect number: a number n is said to be 2-*hyperperfect* if it can be written as $n = 1 + 2 \sum_{\substack{d|n \\ 1<d<n}} d$, which is equivalent to the condition $2\sigma(n) = 3n + 1$; the sequence of numbers satisfying this property begins as follows: 21, 2 133, 19 521, 176 661, 129 127 041, ...; more generally, a number n is said to be *hyperperfect* if there exists a positive integer k such that

$$n = 1 + k \sum_{\substack{d|n \\ 1<d<n}} d, \tag{1}$$

in which case we also say that n is k-*hyperperfect* [9]; the following table contains some k-hyperperfect numbers along with their factorization:

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[9] A 1-hyperperfect number is simply a perfect number. It is easy to show that relation (1) is equivalent to

$$k\sigma(n) = (k+1)n + (k-1). \tag{2}$$

Also, it is clear that a prime power p^α, with $\alpha \geq 1$, cannot be hyperperfect. Furthermore, it follows immediately from (1) that if n is k-hyperperfect, then $n \equiv 1 \pmod{k}$ and moreover that

$$\sigma(n) = n + 1 + \frac{n-1}{k}. \tag{3}$$

This last relation proves to be an excellent tool to determine if a given integer n is a hyperperfect number and also to construct, using a computer, a list of hyperperfect numbers. Indeed, it follows from (3) that

$$n \text{ is a hyperperfect number} \iff \frac{n-1}{\sigma(n) - n - 1} \text{ is an integer.}$$

It also follows from (3) that the smallest prime factor of such an integer n is larger than k. Indeed, assume that $p|n$ with $p \leq k$. We would then have that n/p is a proper divisor of n, in which case

$$\sigma(n) > n + 1 + \frac{n}{p} \geq n + 1 + \frac{n}{k} > n + 1 + \frac{n-1}{k} = \sigma(n),$$

a contradiction. It follows from this that a hyperperfect number which is not perfect is odd.

On the other hand, if n is a square-free k-hyperperfect number, then k must be even. Assume the contrary, that is that k is odd. As we just saw, n must be odd, unless $k = 1$, in which case n would be perfect and even. But then we would have that $n = 2^{p-1}(2^p - 1)$ for a certain prime number $p \geq 3$, in which case n would not be square-free. We therefore have that n is odd. Now, because of (2), we have

$$k\sigma(n) = 2\left(\frac{k+1}{2}n + \frac{k-1}{2}\right). \tag{4}$$

If $k \equiv 1 \pmod{4}$, then it follows from (4) that

$$k\sigma(n) = 2\,(\text{odd} + \text{even}) = 2 \times \text{odd},$$

while if $k \equiv 3 \pmod{4}$, then

$$k\sigma(n) = 2\,(\text{even} + \text{odd}) = 2 \times \text{odd},$$

which means that $2\|\sigma(n)$, in which case n is prime, since n is square-free.

2-hyperperfect	21	$= 3 \cdot 7$
	2 133	$= 3^3 \cdot 79$
	19 521	$= 3^4 \cdot 241$
	176 661	$= 3^5 \cdot 727$
	129 127 041	$= 3^8 \cdot 19\,681$
	328 256 967 373 616 371 221	$= 3^{21} \cdot 31381059607$
3-hyperperfect	325	$= 5^2 \cdot 13$
4-hyperperfect	1 950 625	$= 5^4 \cdot 3\,121$
	1 220 640 625	$= 5^6 \cdot 78\,121$
	186 264 514 898 681 640 625	$= 5^{14} \cdot 30\,517\,578\,121$
6-hyperperfect	301	$= 7 \cdot 43$
	16 513	$= 7^2 \cdot 337$
	60 110 701	$= 7^2 \cdot 383 \cdot 3203$
	1 977 225 901	$= 7^5 \cdot 117\,643$
	2 733 834 545 701	$= 7^4 \cdot 30893 \cdot 36857$
	232 630 479 398 401	$= 7^8 \cdot 40353601$
10-hyperperfect	159 841	$= 11^2 \cdot 1\,321$
11-hyperperfect	10 693	$= 17^2 \cdot 37$
12-hyperperfect	697	$= 17 \cdot 41$
	2 041	$= 13 \cdot 157$
	1 570 153	$= 13 \cdot 269 \cdot 449$
	62 722 153	$= 13^3 \cdot 28\,549$
	10 604 156 641	$= 13^4 \cdot 371\,281$
	13 544 168 521	$= 13^2 \cdot 2347 \cdot 34147$
	1 792 155 938 521	$= 13^5 \cdot 4\,826\,797$

- the number of two digit prime numbers; if we let $C(k)$ stand for the number of k digit prime numbers, then $C(1) = 4$, $C(2) = 21$, $C(3) = 143$, $C(4) = 1\,061$, $C(5) = 8\,363$, $C(6) = 68\,906$, $C(7) = 586\,081$, $C(8) = 5\,096\,876$, $C(9) = 45\,086\,079$, $C(10) = 404\,204\,977$, $C(11) = 3\,663\,002\,302$ and $C(12) = 33\,489\,857\,205$.

22

- the smallest Smith number: a composite number is said to be a *Smith number* if the sum of its digits is equal to the sum of the digits of its distinct prime factors: here $22 = 2 \cdot 11$ and $2 + 2 = 4 = 2 + 1 + 1$ (see U. Dudley [72]).

23

- the prime number which appears the most often as the fifth prime factor of an integer (see the number 199);

- one of the two numbers (the other one being 239) which cannot be written as the sum of less than nine cubes (of non negative integers): here $23 = 2 \cdot 2^3 + 7 \cdot 1^3$ (L.E. Dickson [66]);

- the second number n (and possibly the largest) such that n^3+1 is a powerful number (a number is said to be *powerful* (or *squarefull*) if $p|n$ implies that $p^2|n$); the smallest number satisfying this property[10] is $n=2$;

- one of the nine known numbers k such that $\underbrace{11\ldots1}_{k}$ is prime (see the number 19);

- the largest number which cannot be written as the sum of two non square-free numbers (see the number 933 for a more general problem).

24

- the only number $n>1$ such that $1^2+2^2+\ldots+n^2$ is a perfect square (E. Lucas, 1873) (see the number 70);

- the smallest number m such that equation $\sigma(x)=m$ has[11] exactly three solutions, namely 14, 15 and 23;

- the sixth number n such that $\tau(n)=\phi(n)$ (see the number 8);

- the smallest solution of $\sigma_2(n)=\sigma_2(n+2)$ (see the number 1 079);

- the smallest number with at least two digits, having all its digits different from 1 and 0, and whose sum of digits, as well as the product of its digits, divides n: the sequence of numbers satisfying this property begins as follows: 24, 36, 224, 432, 624, 735, 2232, 3276, 4224, 6624, 23328, 32832, 33264, 34272, 34992, 42336, 42624, 43632, 73332, 82944, 83232, 92232, 93744, ...

25

- the only odd perfect square $\neq 1$ which is not the sum of three perfect squares $\neq 0$ (see E. Grosswald [99], Chapter 3);

- the only perfect square which when increased by 2 yields a cube: $5^2+2=3^3$;

- the number of prime numbers <100.

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[10] One can easily prove that if the *abc* Conjecture is true, then there is only a finite number of numbers satisfying this property.

[11] K. Ford & S. Konyagin [82] proved a conjecture of Sierpinski according to which, for each $k\geq 2$, there exists a number m such that equation $\sigma(x)=m$ has exactly k solutions x. Later, K. Ford [83] proved that this result is also valid for the Euler ϕ function; moreover, this time, the proof also reveals that for each $k\geq 2$, there exist infinitely many m's such that $\phi(x)=m$ has exactly k solutions in x.

26

- the smallest number which is not a palindrome, but whose square is a palindrome; a *palindrome* is a number which reads the same way from the left as from the right; the first ten numbers n satisfying this property are listed below:

n	n^2
26	676
264	69696
307	94249
836	698896
2285	5221225

n	n^2
2636	6948496
22865	522808225
24846	617323716
30693	942060249
798644	637832238736

- the smallest solution of $\sigma(n) = \sigma(n + 15)$; it is mentioned in R.K. Guy [101], B13, that Mientka & Vogt could only find two solutions to this equation, namely 26 and 62: there are at least seven others, namely 20 840 574, 25 741 470, 60 765 690, 102 435 795, 277 471 467, 361 466 454 and 464 465 910.

27

- the smallest number n such that n and $n + 1$ each have exactly three prime factors counting their multiplicity: $27 = 3^3$ and $28 = 2^2 \cdot 7$ (see the number 135 for the general problem with k prime factors instead of only three).

28

- the only even perfect number of the form $a^n + b^n$, with $n \geq 2$ and $(a, b) = 1$: in fact, $28 = 1^3 + 3^3$ (T.N. Sinha [187]).

29

- the smallest prime number $p > 2$ such that $41^{p-1} \equiv 1 \pmod{p^2}$: the only prime numbers $p < 2^{32}$ satisfying this congruence are 2, 29, 1 025 273 and 138 200 401 (see Ribenboim [169], p. 347).

30

- the smallest Giuga number: we say that a composite number n is a *Giuga number* if $\displaystyle\sum_{p|n} \frac{1}{p} - \prod_{p|n} \frac{1}{p}$ is a positive integer: if we could find a number n which is both a Giuga number and a Carmichael number (which is most unlikely!), we would then have found a composite number n satisfying the congruence

$$1^{n-1} + 2^{n-1} + \ldots + (n-1)^{n-1} \equiv -1 \pmod{n}$$

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