Preface

Why did I write this book? I am frequently struck by the fact that students with an undergraduate degree in mathematics know so little about the interrelations between various parts of mathematics. Even those at the start of graduate school so suffer. They may know a lot of algebra and analysis, for example, but few of them have a clue that there are bridges that connect these two subjects and that each affects the other. This book is an attempt to address this situation.

Having a broad view of mathematics is an advantage whether you are a high school teacher, an industrial practitioner, a professor at a liberal arts college, or a research mathematician. Everyone, including the author, has much to learn about the interconnections between various parts of mathematics; though those engaged in exploring the boundaries of mathematics seem to eventually discover several of these connections, at least those associated with their own research. I still remember my delight, as a young assistant professor, at discovering the true test for the nature of a critical point of a function of two variables. That delight turned to a feeling that my education was at fault and then to the realization that I was one of many in the same boat going upstream in the river of mathematics.

This book presents material for a senior-level course for mathematics majors, including those who intend to become school teachers. Most chapters explore relations between different parts of mathematics. The chapters are reasonably self-contained, but some require more sophistication than others. In fact *anyone who examines this book will discover that it is far from homogeneous either in its content or in its demands on the reader in both background and effort. That's intentional and is essentially required by the the great variation in undergraduate preparation.*

Each chapter starts at a point where I think there are a fair number of seniors. In a certain sense this was the most difficult part of writing this book — determining where to start so that most students would find it accessible. In standard courses it is easy to decide where to start. Here I feel that wherever I start I am certain to be ahead of some and behind others. In other words, every possible starting point is a compromise. *This places a greater onus on the teacher to add material or delete sections so as to tailor the presentation to the students in the room.*

To help a little, there is an Appendix with seven sections, arranged not in the order in which they are needed in the book but in a quasi-logical development. The purpose of the Appendix is not to act as a course on these subjects, but to set a starting point, bridge gaps, act as a handy reference for this material, and help increase the comfort level of the students. In fact most of the Appendix covers material only needed as background for Chapter 6 on Modules, which I think is the most difficult in the book. If that chapter is not to be part of the course, there is little need to worry about it.

The first two sections of the Appendix are on groups and rings; just a taste of these topics is provided. In fact Chapter 6 develops a fair amount of ring theory, though only what is needed for its objectives. In the Appendix I tried to be a little more careful in discussing quotient groups and rings as even those students who have studied algebra seem uncomfortable with these concepts. I didn't want a student to avoid a chapter like the one on modules just because they hadn't encountered groups and rings, when all they need is the barest of familiarity with such concepts. Starting in §A.3, I discuss vector spaces over an arbitrary field, which is mildly dependent on the material on rings. Nothing deep is required to digest this as such concepts as linear independence using a field are the same as that idea for a vector space over the real numbers. The idea here is again to be thinking ahead to Chapter 6 where I want to prove the Artin-Wedderburn theorem, and this requires the idea of a vector space over a division ring. I felt the transition from fields to division rings was a bit less abrupt than starting with a linear space over the real numbers.

§A.4 deals with linear transformations, again on vector spaces over an arbitrary field. This can be regarded as being sure that all readers have easy access to some basic facts concerning linear transformations and, at least as important, the examples that are used repeatedly in Chapters 3 through 6. §A.5 discusses lattices, though only the definition and some examples are presented. This language is used in discussing invariant subspaces of a linear transformation. The final two sections of the Appendix state results on triangular representation of a linear transformation over a finite-dimensional vector space over the real or complex numbers. This is used in Chapter 5 on Matrices and Topology. No proofs are presented, only the statements of the results and references to Axler [1996] for the proofs. If Chapter 6 is covered, then the student will see Jordan forms, which are a special case of the triangular forms.

It is inevitable that some who look at this material will think I have misjudged the situation. I still hope they will be willing to present the mathematics in a way that makes it accessible to their students. In fact, that is what I think teachers are for.

Take, for example, linear algebra. What does the typical undergraduate know? If they only had a single course on the subject, the answer is likely that there are huge gaps. That is a course for which, because it is taken by such a large, varied audience, few texts do a job that mathematicians find exciting. I hope most students facing this book will have had a second course in that subject as Chapters 3 through 6 depend heavily on it. This is why I included Chapter 3 to establish a starting point. In fact, Chapters 3, 4, and 6 together with the relevant parts of the Appendix could constitute a second course in linear algebra. It misses a few things one ordinarily covers there, but in the end the important theorems from that course (for example, the Spectral Theorem and Jordan forms) are covered, though from a more advanced point of view.

In fact, this points out something else about this book. There is a lack of balance in that linear algebra surfaces more than any other topic. There are three reasons for this: linear algebra is a subject that every undergraduate has seen, it affords many opportunities for connections, and it is a topic that hardly any undergraduate understands. Of all the courses in the undergraduate curriculum, linear algebra seems to have the shortest half-life. Even good students seem to forget it in less time than they took to learn it. That's a fact. It was true when I was an undergraduate and it remains so today. I think the problem is the sterility with which linear algebra is usually presented. I hope that by covering at least one of the chapters that uses linear algebra, the reader will be prompted to review the subject, appreciate it more fully, and retain the material longer. In fact, a lot of this book might be subtitled, "Variations on a Theme of Linear Algebra."

Some Advice to the Teacher

The material in the book falls into different categories. Some chapters, such as Chapter 5 on Matrices and Topology, seek to show directly how material from two different courses in the undergraduate curriculum are related. Topological questions are asked about sets of matrices, and these questions need linear algebra for the answer. I like this chapter, but it may be that if you teach it, you will have to ask your students to accept some mathematical results, like putting matrices in triangular form, without their ever having seen a proof. That is somewhat counter to the prevailing mathematical culture, but certainly not a violation of what happens in most sciences. I see nothing wrong with this, and, indeed, it might inspire some students to try to learn the proofs of those assumed results.

Other chapters, such as Chapter 2 on Regular Polyhedra, present a beautiful piece of mathematics that, while accessible to undergraduates, seldom finds its way into the undergraduate curriculum. Since this is shown using the Euler characteristic, I think you can still say it gives a relationship between different areas. In fact, the theme of connections is renewed when the Euler characteristic is used to discuss map coloring and tessellations of the plane.

Chapter 1 on Trisecting Angles is, for many, the quintessential instance of the combination of two areas of mathematics — algebra and geometry. This is frequently seen in standard courses on algebra but only in an abbreviated manner. I was quite surprised that the solution of the trisection problem was so accessible. I had been raised to think of it as a consequence of Galois theory. Anyone who knows Galois theory will certainly see its elements in Chapter 1 but nothing close to its full force is required. (Actually Chapter 1 as well as Chapter 2 require minimal mathematical sophistication and background and in a simplified form could be made accessible to bright freshmen. It is, at least partly, why I placed them at the beginning of the book.)

Another topic, like Chapter 4 on the Spectral Theorem, takes a result in the undergraduate curriculum and looks at it from several different points of view. In this case we look at the Spectral Theorem as the diagonalization of hermitian matrices, as a characterization of the unitary equivalence classes of hermitian linear transformations, and as the Principal Axis Theorem for quadratic forms. We then return to the connections theme and apply this result to a study of the critical points of a function of several variables. On the other hand, Chapter 6 on Modules stays within algebra but shows how a discussion of an abstract concept such as a module leads to more concrete results like the Spectral Theorem, Jordan forms for matrices, and the structure of finitely generated abelian groups.

There are many other connections in mathematics that could be made accessible to the same audience. For example, many things in applied mathematics deserve a place in this book; but I lack the expertise and perspective to do them justice and be anything more than a scribe. I decided to leave them until their champion appears.

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No chapter by itself contains enough material for a single course. I could see a course using this book to cover between two and four chapters, depending on the backgrounds of the students and the chapters chosen. If any of Chapters 4 through 6 are covered, then the students have to understand

Chapter 3. Individual instructors should feel free to pick a combination of chapters that best suits their students and their personal inclinations. I could also see an instructor using a single chapter as an enrichment for a standard course in the curriculum. For example, I could also see a course where Chapter 5 is covered while simultaneously teaching the students about metric spaces. Another way I have used this material is as a source for reading courses for undergraduates.

Teaching this type of course is a challenge and a pleasure. The pleasure arises since you can let your tastes dictate the topics and the pace. This is a course where how much you cover is not so important as imparting a perspective and making a point. The challenge arises with the varied students who are likely to be seated in front of you. Unless you are at a small school where almost all the students follow the same path in mathematics, fashioning a course like the one this book was meant to support is going to be more work than teaching a course in one of the usual topics. *I think it inevitable that when you use a book like this one and your goal is the same as the book's and you have students with the same diverse backgrounds as I am used to seeing, then you will have to work a little harder than normal to keep the class together.*

Nevertheless, if you enjoy teaching, it will be rewarding. It is the teacher's role to try to fill in gaps, prod students to push themselves, and adjust the pace of the material to suit the audience.

The object in writing this book was not to present the material, but to teach the student. There are things I do here that I would never do in a monograph. For example, some things are repeated; partially this is done to increase the independence of the chapters, but also repetition is frequently helpful and instructive. In addition, because this book is directed at undergraduates, I wanted to teach them how to learn mathematics. This is reflected in many ways. Some are subtle like the level of detail, talk about intuition, or what I think of as encouragement. Others are more blatant like the frequent insertion into the text of "(Why?)" and "(Verify.)" I also leave many details and routine arguments to the reader; there is no better way to fix ideas in the brain than to carry these out.

Advice to the Student

Here are a few pointers for reading this book — or any mathematics book. First, read with a writing implement and paper nearby. Just reading the words will not suffice; you must fill in details and for that you must write. There is some reading of mathematics where you are only interested in getting the "big picture." But while you are a student at the level at which this book is directed, you should be reading and understanding every detail. (And I do mean every.) You need to develop the skill of reading mathematics.

Read every word. That means pronounce it in your head. This is the opposite of what you should do when you read a novel, but it is the only way I know to learn reading mathematics. In particular I frequently sprinkle throughout the text the parenthetical remarks "Why?" and "Verify." That means I think it important that you heed that question/command. You will frequently see the phrase, "the details are left to the reader." Supply those details. I do not do this because I am lazy but I think what is required to fill in the details is routine and within your reach. You can think of it as a speed bump in your reading or as a test of your comprehension. If you cannot complete the argument, you have missed something and should put on the brakes and back up to the previous result. I certainly want to give you a view of the whole forest, but I also want you to become familiar with the individual trees right down to the patterns of the bark.

You will also see many examples. I have said on more than one occasion that "Mathematics is a collection of examples." Without examples, the theory is vacuous. In fact, the way mathematical concepts have come to be is the observation that several examples and arguments have a commonality and that it is worthwhile to single out the essential ingredients. Results are statements about a collection of examples.

Then there are the exercises. They run the gamut from the routine to the challenging. Spending time on exercises builds your ability to understand and do mathematics. If you are having trouble solving a particular exercise, follow the standard advice: add a reasonable additional hypothesis and see if you can then solve it. Try to construct a counterexample to the exercise and see why it cannot be done.

Using Wikopedia is not frowned on, certainly not by me. In fact, you will see some references in the text to Wikopedia and other web sites. Hopefully they will still exist when the book hits the street. The mathematics topics on the web are frequently well done but sometimes too shallow for you. So if you come across some topic that is a little fuzzy for you, go there and see what it says. It may help or not. If not, try going to your teacher.

Finally there are a few historical notes. I think they are interesting and hope you do too. On some topics I found the history opaque. For example the development of the Spectral Theorem for hermitian matrices is a historical mystery to me and maybe deserves a scholar's effort. Again feel free to browse the web for additional historical notes, though like all that is there, you might want to cross check it with a source that has been subjected to peer review — like a book.

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It goes without saying that the final responsibility for the text is mine. This was a new venture for me, in more than one sense of the word, and I welcome feedback.

Enjoy!

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