Preface

The $3x+1$ problem, also known as the Collatz problem, is a notorious unsolved problem in arithmetic. Consider the operation on positive integers x given by: if x is odd, multiply it by 3 and add 1; while if x is even, divide it by 2. The $3x+1$ problem asks whether, starting from any positive integer x, repeating this operation over and over will eventually reach the number 1. The answer appears to be "yes" for all such x but this has never been proved. Despite its simple appearance, this problem is believed to be extraordinarily difficult.

A goal of this book is to report on what is known about the problem. It is divided into five parts. This book contains two introductory papers on the problem, three survey papers on the problem's connections to various fields, two papers devoted to stochastic models and computational results for the problem, six reprinted papers of historical interest, and a paper which giving an annotated bibliography of work on this problem up to 2000. We now describe in more detail the papers appearing in this volume.

PART I. Overview and Introduction. Part I of this volume contains two introductory papers on the $3x+1$ problem.

(1) Jeffrey C. Lagarias, *The* $3x+1$ *problem: an overview* ([**15**]).

This paper gives a brief review of the $3x+1$ problem, its history, its connection to various fields of mathematics, and its current status. It discusses generalizations of the problem, and describes areas for mathematical research that it impacts. It summarizes current "world record" on various aspects of the problem. It formulates some directions for future research. Finally it discusses whether or not this problem is a good mathematical problem.

(2) Jeffrey C. Lagarias, *The* $3x+1$ *problem and its generalizations*, American Math. Monthly **92** (1985), 3–23 ([**14**]).

This paper describes the $3x+1$ problem and related problems. It presents, with proofs, basic results on the behavior of iterates modulo powers of 2, and shows that most integers n iterate to some value less than n. It formulates basic conjectures on trajectories and cycles, all of them still unsolved. It focuses on number theoretic aspects of the problem. Although this paper only covers work through 1984, it is still up to date as an introduction to the basic features of the problem. This reprinted version includes minor corrections and updates the reference citations.

PART II. Three Survey Papers. Part II of this volume presents three current survey papers on aspects of work on the $3x + 1$ problem.

(3) Marc Chamberland, *A* $3x+1$ *survey: number theory and dynamical systems* ([**3**]).

This paper reports on research progress in various directions on the $3x+1$ problem covering the period 1985 through 2007. It includes number theory results and dynamical systems results, as well as other connections. It emphasizes the problem viewed as a dynamical system, and discusses generalizations to dynamical systems on larger spaces. This paper is a revised and extended version of the author's earlier survey paper [**2**], written in the Catalan language.

(4) Keith R. Matthews, *Generalized* $3x + 1$ *mappings: Markov chains and ergodic theory* ([**19**]).

This paper summarizes work on Markov chain models for iterating generalized $3x + 1$ maps, much of it due to the author. These models supply heuristics for the behavior of iteration of a general class of functions that include the $3x + 1$ function. It contains many interesting examples whose limiting behavior under iteration is not understood, even on a conjectural level. This is a potentially fruitful area for further research.

(5) Maurice Margenstern and Pascal Michel, *Generalized* $3x + 1$ *functions and the theory of computation* ([**18**]).

This paper surveys the appearance of $(3x + 1)-$like functions in mathematical logic and in the theory of computation. In 1972 John Conway [**5**] exhibited a $3x + 1$-like function that gives an undecidable computational problem, and he later ([**6**]) expanded on this construction to show how to formulate any computer program using such functions, presented as a programming language FRACTRAN. Other $3x + 1$-like functions have been used to show that various "small" Turing machines, those with few states and small alphabets, which are not known to be universal computers, nevertheless can encode some apparently difficult problems.

PART III. Stochastic Modelling and Computation Papers. Part III of this volume presents two papers on mathematical modelling and empirical results from large computations.

(6) Alex Kontorovich and Jeffrey C. Lagarias, *Stochastic models for the* $3x + 1$ *problem and related problems* ([**13**]).

This paper reports on various stochastic models for the behavior of $3x + 1$ iterates and for comparison, results on iterates for the $5x + 1$ function. A remarkable feature of the $3x + 1$ map is that, although the iteration is deterministic, the best models for the behavior of the iteration are probabilistic. Such stochastic models make predictions about the behavior for iterating a "generic" integer, and also make predictions about the extreme behavior of some integers that may be observed. These models include random walk models for forward iteration, branching processes and branching random walk models for backwards iteration. A third set of branching process models can model the growth of $3x+1$ trees, viewed 3-adically.

(7) Tomas Olivera e Silva, *Empirical verification of the* $3x + 1$ *conjecture and related conjectures* ([**20**]).

This paper reports on the latest computations of the $3x + 1$ problem, and some related functions. In particular the $3x + 1$ conjecture is verified for all $n \leq 5.764 \times 10^{18}$. Results of the computations include tests of some of the predictions of the stochastic models above. It is an interesting challenge to write efficient progams to verify the $3x + 1$ Conjecture to some bound and also collect statistics on the conjecture.

PART IV. Reprinted Early Papers. Part IV of this volume presents six early papers of historical interest, in chronological order. These papers are short, easy to read, and most are hard to obtain in their original source. We provide an editorial commentary after each paper, including additional references and biographical information.

(8) H. S. M. Coxeter, *Cyclic sequences and frieze patterns: The Fourth Felix Behrend Memorial Lecture*, Vinculum **8** (1971), 4–7 ([**7**]).

This paper is the written version of a 1970 lecture, which was published in 1971 in Vinculum, the journal of the Mathematical Association of Victoria (Australia). To my knowledge it is the earliest published paper that explicitly states the $3x+1$ problem. It presents the problem at the end of the lecture as "mathematical gossip." Coxeter offers a $50.00 prize for its solution. The main subject of the paper, cyclic sequences and frieze patterns, is of interest in its own right.

(9) John H. Conway, *Unpredictable iterations*, in: *Proc. 1972 Number Theory Conference (Univ. Colorado, Boulder, Colo., 1972)*, Univ. Colorado, Boulder, Colo. 1972, pp. 49–52 ([**5**]).

This 1972 paper, from the proceedings of a number theory conference held at the University of Colorado, shows that a generalization of the $3x + 1$ problem is undecidable. Conway later used this encoding to design a computer language FRACTRAN for universal computation using multiplication of fractions, see paper (13) below.

(10) C. J. Everett, *Iteration of the number theoretic function* $f(2n) = n$, $f(2n + 1) = 3n + 2$, Advances in Math. **25** (1977), 42–45. ([**9**]).

This 1977 paper gives an elegant proof of a basic result showing that almost positive integers n iterate to a smaller value under action iteration of the $3x+1$ function. A similar result was independently obtained in 1976 by Riho Terras [**21**], [**22**].

(11) Richard K. Guy, *Don't try to solve these problems!*, American Math. Monthly **90** (1983), 35–41 ([**10**]).

This 1983 paper, written for the Unsolved Problems column of the American Mathematical Monthly, presents a potpourri of $3x + 1$-like problems, including the original problem. True to its name, so far none of the four problems formulated there have been solved.

(12) Lothar Collatz, *On the motivation and origin of the* $(3n+1)$ *problem* (Chinese), J. Qufu Normal University, Natural Science Edition [Qufu shi fan da xue xue bao] **12** (1986), No. 3, 9–11 ([**4**]).

This 1986 paper, written in Chinese, is the only paper of Lothar Collatz that discusses his work on the $3x+1$ problem. It is based on a talk that Collatz gave at Qufu Normal University, Qufu, Shandong, China. Here we present an English translation of this paper, using Collatz's original illustrations.

(13) John H. Conway, *FRACTRAN: A simple universal programming language for arithmetic*, In: *Open Problems in Communication and Computation* (T. M. Cover and B. Gopinath, Eds.), Springer-Verlag: New York 1987, pp. 3-27 ([**6**]).

This 1987 paper of Conway expands on his 1972 paper to show how to encode any computational problem in terms of iteration of a suitable $3x+1$-like function. The programming language name FRACTRAN is a pun on FORTRAN (The IBM Mathematical Formula Translating System). This is not the only pun in this paper.

PART V. Annotated Bibliography. Part V of this volume an an annotated bibliography of work on the $3x+1$ problem and related iteration problems, from 1963-1999.

(14) Jeffrey C. Lagarias, *The* $3x+1$ *problem: An annotated bibliography (1963-1999)* ([**16**]).

This bibliography attempts to be relatively complete over the period cited. It includes a number of papers from the "prehistory" of the problem, in the 1960's. It also covers many papers appearing in unusual places, not covered by Mathematical Reviews or Zentralblatt für Mathematik. It groups papers on the problem into ten year subintervals. The growth of the number of papers in these time intervals, which total 8, 34, 52 and 103 papers, respectively, show increasing effort devoted to the $3x+1$ problem and generalizations. A follow-up bibliography, currently covering the period 2000–2009 ([**17**]) is posted on the math arXiv.

Book Title: The Ultimate Challenge. The results known about the $3x+1$ problem strongly suggest that it does not fit in the scope of classical "structural" mathematics. Instead it seems to lie in a wilderness between the well-organized part of mathematical knowledge, typified by the subjects covered in the volumes of Bourbaki, and the boundary of undecidable problems, those problems that can encode the action of a universal computer. The title does not assert the problem is "ultimate" in its importance. Rather, "ultimate challenge" refers to the contrast between the simplicity of the statement of the problem and the apparent difficulty (perhaps impossibility) of resolving the problem. The papers in this volume give ample warning that the problem shows no sign of being solvable at present. Remember, not all challenges need to be accepted!

Epigraphs: References. The statement of G. C. J. Jacobi is taken from a letter to Legendre written in 1830, published in 1875 in Borchardt [**1**, p. 272]

> Il est vrai que M. Fourier avait l'opinion que le but principal de mathématiques était l'utitlité publique et l'explications des phénomès naturels; mais un philosophe comme lui aurait dû savoir que le but unique de la science, c'est l'honneur de l'esprit humain, et que sous ce titre, une question des nombres vaut autant qu'une question du système du monde.

The statement of D. Hilbert is taken from a 1931 paper on foundations on mathematics (Hilbert [**12**, p. 486]).

> Es ist schon an sich merkwürdig und philosophich bedeutsam, dass die ersten und einfachsten Fragen úber die Zahlen $1, 2, 3, ...$ so tieflegende Schwierigkeiten bieten. Diese Schwierigkeiten müssen überwunden werden.

The statement of P. Erdős was made in conversation, sometime after publication of the 1985 survey paper ([**8**]).

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