

---

# Contents

Preface	xv
Acknowledgments	xxi
To the Reader	xxv
Some Conventions in this Book	xxxix
<b>Part 1. Whole Numbers</b>	
Part Preview	3
Chapter 1. Place Value	5
1.1. How to Count	6
1.2. Place Value	19
1.3. The Use of Symbolic Notation	21
1.4. The Number Line	22
1.5. Comparing Numbers (Beginning)	25
1.6. Multiplication and the Expanded Form of a Number	27
1.7. All about Zero	32
1.8. The Hindu-Arabic Numeral System	33
Exercises	35
Chapter 2. The Basic Laws of Operations	39
2.1. The Equal Sign	39
2.2. The Associative and Commutative Laws of +	40
2.3. The Associative and Commutative Laws of $\times$	43

---

2.4. The Distributive Law	44
2.5. Comparing Numbers (Conclusion)	48
2.6. An Application of the Associative and Commutative Laws of Addition	51
Exercises	53
Chapter 3. The Standard Algorithms	57
Chapter 4. The Addition Algorithm	61
4.1. The Basic Idea of the Algorithm	61
4.2. The Addition Algorithm and Its Explanation	62
4.3. Essential Remarks on the Addition Algorithm	65
Exercises	69
Chapter 5. The Subtraction Algorithm	71
5.1. Definition of Subtraction	71
5.2. The Subtraction Algorithm	73
5.3. Explanation of the Algorithm	75
5.4. How to Use the Number Line	79
5.5. A Special Algorithm	81
5.6. A Property of Subtraction	82
Exercises	83
Chapter 6. The Multiplication Algorithm	85
6.1. The Algorithm	85
6.2. The Explanation	87
Exercises	92
Chapter 7. The Long Division Algorithm	95
7.1. Multiplication as Division	97
7.2. Division-with-Remainder	103
7.3. The Algorithm	107
7.4. A Mathematical Explanation (Preliminary)	110
7.5. A Mathematical Explanation (Final)	117
7.6. Essential Remarks on the Long Division Algorithm	119
Exercises	123
Chapter 8. The Number Line and the Four Operations Revisited	127
8.1. The Number Line Redux, and Addition and Subtraction	127
8.2. Importance of the Unit	129

---

8.3. Multiplication	131
8.4. Division	132
8.5. A Short History of the Concept of Multiplication	134
Chapter 9. What Is a Number?	135
Chapter 10. Some Comments on Estimation	139
10.1. Rounding	140
10.2. Absolute and Relative Errors	144
10.3. Why Make Estimates?	147
10.4. A Short History of the Meter	150
Exercises	152
Chapter 11. Numbers in Base $b$	155
11.1. Basic Definitions	155
11.2. The Representation Theorem	158
11.3. Arithmetic in Base 7	161
11.4. Binary Arithmetic	164
Exercises	168
<b>Part 2. Fractions</b>	
Part Preview	173
Chapter 12. Definitions of Fraction and Decimal	177
12.1. Prologue	179
12.2. The Basic Definitions	183
12.3. Decimals	187
12.4. Importance of the Unit	188
12.5. The Area Model	190
12.6. Locating Fractions on the Number Line	195
12.7. Issues to Consider	196
Exercises	199
Chapter 13. Equivalent Fractions and FFFP	203
13.1. Theorem on Equivalent Fractions (Cancellation Law)	204
13.2. Applications to Decimals	207
13.3. Proof of Theorem 13.1	209
13.4. FFFP	212
13.5. The Cross-Multiplication Algorithm	214
13.6. Why FFFP?	216

---

Exercises	218
Chapter 14. Addition of Fractions and Decimals	221
14.1. Definition of Addition and Immediate Consequences	222
14.2. Addition of Decimals	224
14.3. Mixed Numbers	225
14.4. Refinements of the Addition Formula	227
14.5. Comments on the Use of Calculators	229
14.6. A Noteworthy Example of Adding Fractions	230
Exercises	232
Chapter 15. Equivalent Fractions: Further Applications	235
15.1. A Different View of a Fraction	236
15.2. A New Look at Whole Number Divisions	238
15.3. Comparing Fractions	239
15.4. The Concept of $\frac{m}{n}$ of $\frac{k}{\ell}$	245
Exercises	250
Chapter 16. Subtraction of Fractions and Decimals	253
16.1. Subtraction of Fractions and Decimals	253
16.2. Inequalities	256
Exercises	258
Chapter 17. Multiplication of Fractions and Decimals	261
17.1. The Definition and the Product Formula	263
17.2. Immediate Applications of the Product Formula	269
17.3. A Second Interpretation of Fraction Multiplication	272
17.4. Inequalities	278
17.5. Linguistic vs. Mathematical Issues	278
Exercises	280
Chapter 18. Division of Fractions	283
18.1. Informal Overview	283
18.2. The Definition and Invert-and-Multiply	286
18.3. Applications	291
18.4. Comments on the Division of Decimals	296
18.5. Inequalities	302
18.6. False Doctrines	303
Exercises	306

---

Chapter 19. Complex Fractions	309
19.1. The Basic Skills	310
19.2. Why Are Complex Fractions Important?	315
Exercises	317
Chapter 20. Percent	319
20.1. Percent	319
20.2. Relative Error	325
Exercises	328
Chapter 21. Fundamental Assumption of School Mathematics (FASM)	331
Chapter 22. Ratio and Rate	335
22.1. Ratio	336
22.2. Why Ratio?	344
22.3. Rate	344
22.4. Units	348
22.5. Cooperative Work	349
Exercises	355
Chapter 23. Some Interesting Word Problems	357
Exercises	364
Chapter 24. On the Teaching of Fractions in Elementary School	367
<b>Part 3. Rational Numbers</b>	
Chapter 25. The (Two-Sided) Number Line	375
Chapter 26. A Different View of Rational Numbers	379
Chapter 27. Adding and Subtracting Rational Numbers	381
27.1. Definition of Vectors	382
27.2. Vector Addition for Special Vectors	383
27.3. Addition of Rational Numbers	386
27.4. Explicit Computations	387
27.5. Subtraction as Addition	390
Exercises	394
Chapter 28. Adding and Subtracting Rational Numbers Redux	395
28.1. The Assumptions on Addition	396
28.2. The Basic Facts	397
28.3. Explicit Computations	399

---

28.4. Basic Assumptions and Facts, Revisited	400
Exercises	402
Chapter 29. Multiplying Rational Numbers	403
29.1. The Assumptions on Multiplication	404
29.2. The Equality $(-m)(-n) = mn$ for Whole Numbers	405
29.3. Explicit Computations	408
29.4. Some Observations	410
Exercises	412
Chapter 30. Dividing Rational Numbers	413
30.1. Definition of Division and Consequences	413
30.2. Rational Quotients	416
Exercises	420
Chapter 31. Ordering Rational Numbers	421
31.1. Basic Inequalities	421
31.2. Powers of Rational Numbers	425
31.3. Absolute Value	426
Exercises	429
<b>Part 4. Number Theory</b>	
Part Preview	433
Chapter 32. Divisibility Rules	435
32.1. Review of Division-with-Remainder	435
32.2. Generalities about Divisibility	436
32.3. Divisibility Rules	439
Exercises	444
Chapter 33. Primes and Divisors	445
33.1. Definitions of Primes and Divisors	445
33.2. The Sieve of Eratosthenes	448
33.3. Some Theorems and Conjectures about Primes	450
Exercises	454
Chapter 34. The Fundamental Theorem of Arithmetic (FTA)	455
Exercises	461
Chapter 35. The Euclidean Algorithm	463
35.1. Common Divisors and Gcd	463

---

35.2. Gcd as an Integral Linear Combination	466
Exercises	472
Chapter 36. Applications	473
36.1. Gcd and Lcm	473
36.2. Fractions and Decimals	478
36.3. Irrational Numbers	480
36.4. Infinity of Primes	483
Exercises	485
Chapter 37. Pythagorean Triples	487
Exercises	489
<b>Part 5. More on Decimals</b>	
Part Preview	493
Chapter 38. Why Finite Decimals Are Important	495
Chapter 39. Review of Finite Decimals	497
Exercises	504
Chapter 40. Scientific Notation	505
40.1. Comparing Finite Decimals	505
40.2. Scientific Notation	507
Exercises	510
Chapter 41. Decimals	511
41.1. Review of Division-with-Remainder	511
41.2. Decimals and Infinite Decimals	513
41.3. Repeating Decimals	518
Exercises	523
Chapter 42. Decimal Expansions of Fractions	525
42.1. The Theorem	525
42.2. Proof of the Finite Case	528
42.3. Proof of the Repeating Case	529
Exercises	539
Bibliography	541
Index	545