Manifesto

When I realized that I didn't have a clear picture of what I was actually doing as a mathematician, I looked for help on the library shelves. I was surprised and disappointed at how little help I found. What was helpful usually had been written by a fellow mathematician, not by a philosopher.

I have tried, little by little, to see as clearly as possible what I am actually doing—what is my own "mathematical experience," so to speak. Here in this manifesto I state some conclusions I have come to, and point to articles below where these conclusions are elaborated and defended.

...Mathematics isn't a fiction, it's a reality.

...Mathematics is not primarily syntactic deductions from meaningless sentences made out of undefined terms. It is meaningful, understandable, communicable.

...Mathematics is not "out there," in an abstract realm apart from human consciousness or material reality. It is "down here," in our individual minds and in our shared consciousness.

"Certain kinds of ideas (concepts, notions, conceptions, and so forth) have science-like quality. They have the rigidity, the reproducibility, of physical science. They yield reproducible results, independent of particular investigators. Such kinds of ideas are important enough to have a name. Study of the lawful, predictable parts of the physical world has a name, 'physics.' Study of the lawful, predictable parts of the social-conceptual world also has a name, 'mathematics.' A world of ideas exists, created by human beings, existing in their shared consciousness. These ideas have objective properties, in the same sense that material objects have objective properties. The construction of proof and counter-example is one method of verifying the properties of these ideas. This branch of knowledge is called mathematics." (*What is mathematics, really?*, Page 19)

...A mathematical entity is a concept, a shared thought. Once you have acquired it, you have it available, for inspection or manipulation. If you understand it correctly (as a student, or as a professional) your "mental model" of that entity, your personal representative of it, matches those of others who understand it correctly. (As is verified by giving the same answers to test questions.) The concept, the cultural entity, is nothing other than the collection of the mutually congruent personal representatives, the "mental models", possessed by those participating in the mathematical culture.

The distinctive characteristic of mathematical concepts, compared to other shared concepts, is their *near-complete unanimity*. In this respect they are like the experimental facts of an empirical science. All those who have correctly mastered or internalized a mathematical concept (according to the standard judgments carried out by the mathematical community) will agree on the properties of that concept, after due and necessary communication and conversation. This unanimous agreement and acceptance is generally attained by the procedure called "proof"—reasoning on the properties of mathematical entities, based on direct observation of one's own models of such entities. I call it "mathematicians' proof," to recognize its distinctness from the "formal proofs" taught in logic. This distinction is elaborated below in *To Establish New Mathematics, We Use Our Mental Models and Build on Established Mathematics*.

A new branch of learning is accepted and absorbed into established mathematics (as happened to probability theory, years ago, and later to proof theory and model theory in logic, and more recently to information theory and cryptography and network theory), if that new branch has conclusive, compelling arguments accepted by everyone who is qualified, who understands what it is all about, who possesses the concept (See *Definition of Mathematics*).

The individual mental aspect means that once you've mastered a mathematical concept, it's yours—to use, to inspect, to consider, to turn upside down and inside out. That's how mathematical research is done. That's why mathematicians' proof is convincing and compelling. This point is elaborated below in *How Mathematicians Convince Each Other or "The Kingdom of Math is Within You"*.

This notion of the individual or personal mental model of a mathematical concept enables us to clarify an important issue in mathematics education. What do we mean by saying a student "understands" a concept? There is a contrast between "really understanding" and "just going through the motions." Going through the motions—carrying out an algorithm correctly—may be acceptable for a passing grade. Really understanding deserves an A. But what do we mean by "really understanding"? What we really mean, I would argue, is possessing an adequate mental model of the concept. Possessing an adequate mental model is displayed by answering questions the student hasn't seen before, or by being able to "play around" with a concept, to connect it with other concepts or to modify it sensibly. This is sometimes called "learning to think like a mathematician." (This educational aspect of the notion of personal or individual mental model of a mathematical concept is not elaborated in any of the papers collected here, or already published elsewhere. It may become the subject of a future paper.)

This individual mental aspect has two essential sub-aspects. One aspect is consciousness. When I'm thinking about a geometry problem, I'm conscious of my thoughts. But an insight, a previously unseen connection or possibility, can arise in some hidden, obscure way, from my subconscious, my "intuition". Further "down below" are the basis of my consciousness, namely, my electro-chemical brain and nerve processes. When I am thinking about mathematical questions, I have no concern with that deep level. But when I am philosophizing about mathematical practice, I keep it in mind, I recognize it. This helps me to recognize and accept that *mathematical thoughts and ideas actually exist; they are real entities*. They are embodied, they are made possible by the thinker's flesh and blood, by his/her functioning brain and nervous system. We experience our own thoughts in the most direct and immediate way conceivable. At the same time, they are grounded in biological and physical reality. (See *How Mathematicians Convince Each Other or "The Kingdom of Math is Within You"*.)

A thought as experienced directly, subjectively, and a thought as an electro-chemical event in my nervous system, are not two different things. They are two different ways of viewing the same thing, different aspects or versions of a single entity. (See *Mathematical Intuition (Poincaré, Polya, Dewey); To Establish New Mathematics, We Use Our Mental Models and Build on Established Mathematics*; and *How Mathematicians Convince Each Other or The Kingdom of Math is Within You".*)

Mathematical knowledge being an activity of fallible flesh and blood, it is fallible and tentative. Most fallible and tentative when the knowledge is recent and complex, less so when it is ancient and simple.

Mathematicians' proof is the principal way a piece of mathematics becomes part of established mathematics. Its proof is a warrant for asserting it in other proofs. (See *Mathematical Intuition (Poincaré, Polya, Dewey).*) This view connects the philosophy of mathematical practice with the pragmatism of John Dewey.