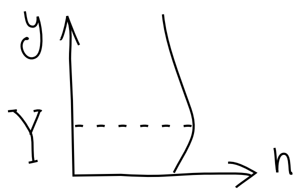
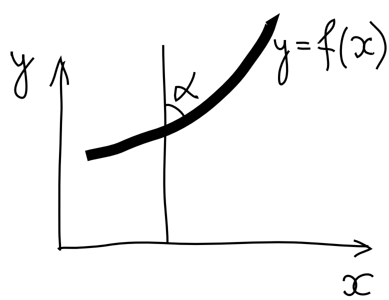
Chapter 10

Mirages

The index of refraction $n(y)$ of air at altitude y over a desert is maximal at a certain altitude Y (at which the air density is maximal: The heat of the desert drives lower layers up, and at high altitude, the density of the atmosphere drops down to zero).

Explain the mirage phenomenon in view of such a behavior of the index of refraction.



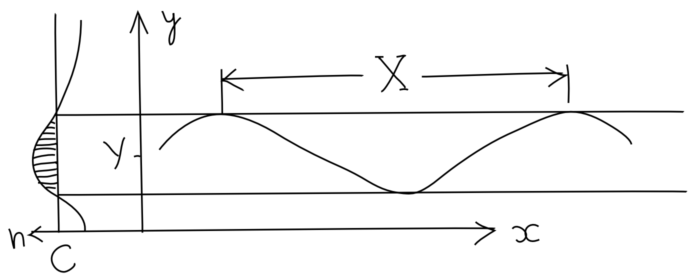


Solution. Let us study the motion $y = f(x)$ of light beams by using the law of refraction $n \sin\alpha = \text{const}$, where α is the angle between the beam and the vertical.

We obtain a (differential) equation for beams of the form

$$\alpha(y) = \arcsin \frac{C}{n(y)}.$$

The parameter C is determined by the choice of the beam under examination. We conclude that a beam (with fixed C close to Y) is entirely contained in the strip where $n(y) \geqslant C$ (and oscillates between its boundaries):



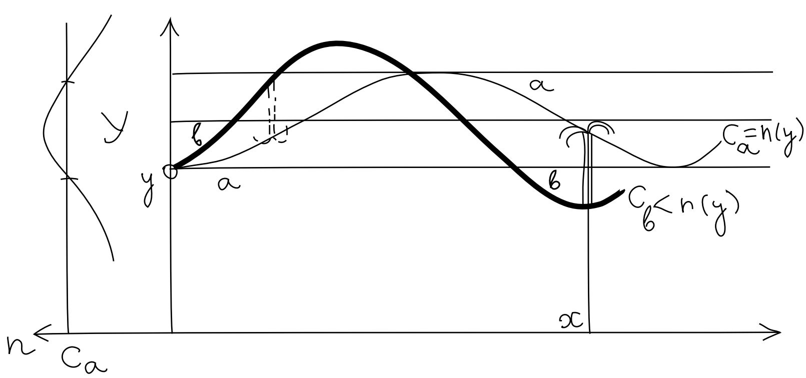
These oscillations render the beam wave-shaped (with wavelength X depending on the constant C).

The value $X(C)$ is finite if C is not a critical value for the index of refraction n: if $dn/dy \neq 0$ at the points where $n(y) = C$.

As the constant C increases to the critical value $n(Y)$ of the index of refraction, the wavelength $X(C)$ grows to infinity, the wave amplitude tends to zero, and the beam propagates along the line $y = Y$.

To understand how the tortuosity of beams affects the images of remote palms, let us look from the point $(x = 0, y)$ at a palm growing at a distance x.

Let us draw rays a and b from the top and the bottom of the palm to the observation point $(x = 0, y)$.



At the observation point $(0, y)$, the ray a, which issues from the top of the palm, is below the ray b, which issues from its bottom. Therefore, the image of the palm turns upside down—that is what the mirage phenomenon is all about.

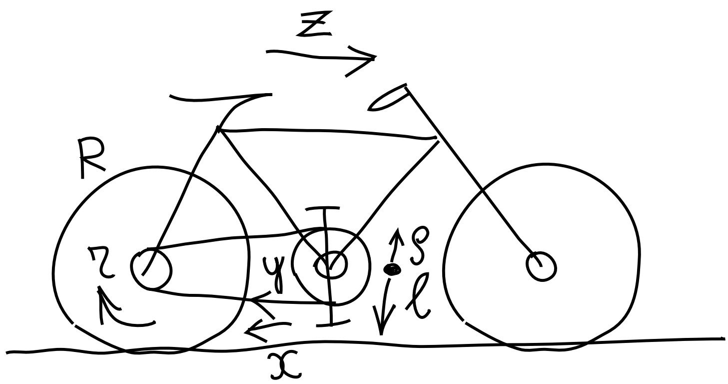
Remark. To comprehend all this, we must clearly understand how the geometry of beams of light is related to the images (of the emitting objects by these beams for the observer).

This relation ("image construction") is explained when lenses are described in high-school physics courses, but only a few students understand this theory. (To see a mirage, it is not necessary to go to a desert: in summer, looking along the platform while awaiting a commuter train, it is easy to see puddles at a distance, although the platform is perfectly dry; noticing this, smart kids come around to the theory described above, but they are few.)

Chapter 13

What Force Drives a Bicycle Forward?

The lower pedal of a bicycle standing still on a horizontal floor is pulled back. Which way does the bicycle go, and in what direction does the pulled back lower pedal move with respect to the floor?



Solution. Let us denote the length of the crank arm (from the pedal to the axle) by l, the radii of the front and rear sprockets (toothed wheels) by ρ and r, and the radius of the rear wheel by R.

Let x be the (backward) displacement of the pedal with respect to the axle. The lowest tooth of the front (and, hence, the rear) sprocket moves back a distance of $y = x(\rho/l)$.

Therefore, the rear wheel turns by an angle such that its point of tangency with the floor covers a distance of

$$z = y\Big(\frac{R}{r}\Big) = x\Big(\frac{\rho}{l}\Big)\Big(\frac{R}{r}\Big).$$

Looking at the bicycle, we estimate the parameter values as

$$l \approx 2\rho, \qquad R \approx 10r.$$

Therefore, the displacement z of the bicycle with respect to the floor is

$$z \approx 5x \qquad \text{(forward!)}.$$

This is the displacement of the axle; the pedal moves backward by x relative to the axle crank arm and forward by $4x$ relative to the floor.

Answer. The bicycle moves forward, and the lower pedal pulled back moves forward as well but 20% less than the whole bicycle.

Remark. It seems surprising that a force directed back (applied to the pedal) forces the bicycle to move forward. But the rotation of the rear wheel creates at its point of tangency with the floor a forward friction force, which wins.

Original Editor's Note. After the first edition of this book was published, some readers correctly noticed that the model considered above is inaccurate.

Corrections were being preliminarily discussed with the author, and it was planned to finalize them before publishing a new edition of the book. The sudden death of Vladimir Igorevich Arnold on June 3, 2010, prevented this.

Considering it wrong to change the original text, we leave the construction of a correct model to the reader. It is assumed that the force is applied to the pedal (rigidly connected with the wheel) by a rider sitting on the bicycle's saddle.