Preface

There are a quite a few excellent topology textbooks written for undergraduate mathematics majors in their junior or senior year of college. Such books usually begin with an introduction to point set topology, followed by geometric topics such as homotopy, covering spaces, knot theory, the classification of surfaces, or some area of applied topology. While not all undergraduate topology texts follow this precise structure, almost all presuppose experience with proofs and exposure to a rigorous definition of continuity. For math departments that do not have a large number of majors or have an inordinate burden of service courses to teach, offering a topology course which presupposes such a background is almost impossible. As a result, many mathematics majors graduate without having the opportunity to take a topology class.

In addition to textbooks written for math majors, there are a number of noteworthy intuitive introductions to topology intended for a wider audience. Such books are great for curious students and for the general public, but can be difficult to teach out of because they rarely include definitions, statements of theorems, or a sufficient number of elementary exercises (without solutions in the back).

What seems to be absent from the available topology books is one that is easy to teach from, includes a wide range of topics, has no prerequisites, and makes no assumptions about the mathematical sophistication or motivation of the reader. Such a book could be the basis for a course that would attract students to the math major who might not be excited by the analytical and algebraic arguments that they typically see in their math courses during the first two years of college.

This book is intended to fill this gap. It is an elementary introduction to geometric topology and its applications to chemistry, molecular biology, and cosmology. It does not assume any particular mathematical or scientific background, sophistication, or even motivation to study abstract mathematics. It is meant to be fun and engaging while at the same time drawing students in to learn about fundamental topological and geometric ideas. Though the book can be read and enjoyed by non-mathematicians, college students, or even eager high school students, it is intended to be used as an undergraduate textbook.

With this in mind, all of the concepts are introduced with explicit definitions and the theorems are clearly stated so that it is easy for an instructor to develop lectures and for students to refer to when doing the homework. Some theorems are proved formally, others are proved intuitively, and for those results that aren't proved, students are told in which advanced course they might expect to see a proof. In addition, each chapter concludes with numerous exercises which are at a level appropriate for students in their first two years of college, and whose solutions are only available to instructors.

In contrast with most topology textbooks, the style of the book is informal and lively. Though all of the definitions and theorems are explicitly stated, they are given in an intuitive rather than a rigorous form. For example, rather than introducing the formal definition of an isotopy, two knots are defined to be equivalent if one can be deformed to the other. This style of presentation allows students to develop intuition about topology and geometry without getting bogged down in technical details. In addition, the topics in the book were chosen for their visual appeal, which is highlighted by the abundance of illustrations throughout. In order to make the text more fun and engaging, early on it introduces a cast of characters who reappear in multiple chapters. This includes the 1-dimensional character A. Dash, the 2-dimensional characters A. Square and B. Triangle, and the 3-dimensional character A. 3D-Girl, each of whom is learning about geometry and topology in order to understand his or her own universe. The characters are illustrated with amusing pictures that entice readers to think about the experience from the character's point of view.

The book is divided into three parts corresponding to the three areas referred to in the title. Part 1, consisting of Chapters 1–8, concerns two and three dimensional universes, though there is a brief foray into the fourth dimension in Chapter 2. The goal of Part 1 is to develop techniques that enable creatures in a given space to visualize possible shapes for their universe, and to use topological and geometric properties to distinguish one such space from another.

The second part of the text, consisting of Chapters 9–11, provides an introduction to knots and links. Building on the ideas developed in Part 1, the emphasis in Part 2 is on deformations of knots and links, and the use of invariants to distinguish inequivalent knots and links. Tricolorability is introduced in Chapter 9 as an invariant which is relatively simple to understand and apply but does not distinguish many knots. Then Chapter 10 surveys a collection of invariants which are not hard to understand but in some cases are hard to compute, and in the exercises we ask the reader to figure out what invariants should be used to distinguish a given pair of knots or links. Finally, Chapter 11 introduces the Kauffman bracket and Jones polynomial, and provides a step-by-step explanation of how to compute them, as well as some theorems that illustrate their power.

The third part of the text, consisting of Chapters 12–15, presents applications of topology and geometry to chemistry and molecular biology. In particular, Chapter 12 compares the concepts of mirror image symmetry from geometric, topological, and chemical viewpoints. It also introduces the idea of using embeddings of graphs in 3-dimensional space as models of non-rigid molecules. Then Chapter 13 presents techniques that can be used to show that some non-rigid molecules are topologically distinct from their mirror images. Chapter 14 explores the topology and geometry of DNA, developing material about rational tangles as a way to model DNA recombination. Finally, Chapter 15 describes topologically complex protein structures, giving examples of knotted and linked proteins, as well as proteins containing non-planar graphs. While Chapter 13 depends on Chapter 12, Chapters 14 and 15 are independent of one another and of Chapters 12 and 13. Thus an instructor can choose which of the applied topics they would like to cover.

The three parts of the book make it usable as a textbook for several different types of courses. For example, Part 1 of the book could be the basis for a seminar for first year college students on a topic such as "Dimensions and the Shape of the Universe." Since the book is written with students in mind, the instructor could give daily reading assignments and then lead the class in a discussion of the material followed by breaking the class into small groups to work through some of the exercises. Such a seminar would also lend itself well to open ended writing projects on diverse topics: Describe a sport that could take place in a 2-dimensional universe. Pick a well-known game (for example chess, checkers, Othello, Hex, or Go), and explain how the strategy for the game would change if it were played on a flat torus or a flat Klein bottle. Design a 4-dimensional version of such a game, and explain how the strategy compares to the usual 2- or 3-dimensional version. Imagine the form of a 4-dimensional person, and explain what we would see if the person passed through our space in different ways. If we had a spaceship that could travel in any direction as fast and far as we would like, what evidence should the spaceship seek to help us figure out the shape of our universe? These are a few of the topics.

The book could serve as the text for a topology course for math majors with or without prerequisites. The instructor could either cover all three parts of the text, or pick and choose topics from each of the three parts and supplement with point set topics from a more traditional topology text. Another option for a course for math majors would be for the instructor to give lectures to introduce each chapter, and then have students take turns giving lectures on each section within a chapter. The sections were intentionally made short to enable students to read and understand a single section at a time.

The text could also be used for an interdisciplinary course linking topology with molecular biology and chemistry. In this case, the course would focus on Part 2 and Part 3 (comprising Chapters 9–15). Part 2 would give students enough background in knot theory to understand the applications to molecular symmetry and the topology of DNA and proteins developed in Part 3. Note that it is not necessary for students to read Part 1 in order to understand Part 2, nor is it necessary for students to have a background in biology or chemistry in order to understand Part 3.

A geometry course for future teachers could also be based on this text. Such courses often consist of an axiomatic approach to Euclidean and non-Euclidean geometry. However, approaching geometry this way can be tedious and may actually kill any natural interest the student has in geometry. By contrast, a course for future teachers based on this text would focus on developing geometric thinking and visual intuition together with problem solving and mathematical reasoning. Such a course would cover Part 1, culminating in Chapter 8 which presents Euclid's five Axioms and alternatives to the Fifth Axiom that are valid on a sphere and a hyperbolic plane. The instructor for such a course would want to present the material slowly in order to emphasize visualization and logical deduction throughout. The nearly 450 illustrations in the text are designed to help students develop their visual intuition, and the many exercises at the end of each chapter would give students necessary practice writing mathematical arguments. Furthermore, the exercises are designed to encourage students to explain their reasoning in complete sentences and integrate illustrations into their explanations whenever possible. This course could culminate with a project such as designing a museum exhibit featuring the student's choice of the ten most important images from the text with an introduction to the exhibit and short explanatory descriptions associated with each image.

The students could also be asked to include some 3-dimensional models constructed themselves.

Finally, because the text is readable by students and is self-contained, the book would be well suited for an independent study or senior year project. Also, because of its lack of prerequisites and its division into three parts and the subdivision of each chapter into short sections, it would work well as the textbook for a four-week winter term or a summer course.

We hope that at least in some small way this book contributes to one or more of the following large goals: it motivates some undergraduates to major in math; it motivates some math majors to study topology; it motivates some math students to read their textbooks; it convinces some future K–12 teachers that geometry is fun; or it convinces some future scientists, administrators, or government officials that even the purest areas of mathematics can have important applications. At the very least, we hope that you and your students enjoy reading it.