Preface

Training has no shortcuts.

Golden State Warriors
Ramp Run video,
October 24, 2012
([GoldenState])

This volume and its companion volume—*Teaching School Mathematics: Algebra* ([Wu-Alg])—address the mathematics that is generally taught in grades 5–9. They are not student texts, however, because they have been written expressly for teachers, especially middle school teachers. These two volumes are designed *not* to show you how mathematics is really just common sense and lots of fun, but to help you teach the mathematics of middle school in a way that meets the minimal standards of human communication. In other words, problems are solved without recourse to tricks or any *ad hoc* sleight-of-hand, every step is explained logically using only concepts and skills already developed, and every concept is clearly defined so that no clever guessing is needed for its understanding. There may be an added bonus in that the mathematical development of these volumes parallels that of the Common Core State Standards for Mathematics ([CCSSM]) for middle school.

These volumes differ from the usual presentations found in standard school textbooks (and professional development materials as well) in substantial ways. First and foremost, the presentations in the standard textbooks, be they traditional or reform, are riddled with mathematical errors, thanks to *Textbook School Mathematics* (**TSM**).[1] While the Table of Contents bears a superficial resemblance to what you normally find in school textbooks and other professional development materials, there are major differences in terms of precision, sequencing, and reasoning. It is hoped that these volumes will lead you to rethink some of this material even if you believe you already know it very well.

[1]This is the name given to the *mathematics* in almost all standard school mathematics textbooks of roughly the past four decades. It is notable for being antithetical to the five principles listed on pages xv ff. A more elaborate discussion of TSM can be found in [Wu2013b] and [Wu2015].

The first major departure from TSM in these volumes is the treatment of fractions and rational numbers. Fractions (and rational numbers) are the backbone of K–12 mathematics and are therefore the centerpiece of not only these two volumes, but also the other volumes written for teachers: [Wu2011a] and [Wu-HighSchool]. Contrary to the prevailing norm in mathematics education, these volumes will ask you to spread the message that:

(1) Fractions are numbers that you can compare to see which is bigger, and can add, subtract, multiply, and divide.
(2) The number line is home for all (real) numbers, including whole numbers, fractions, and rational numbers.
(3) Fractions of a fixed denominator, when viewed as multiples of the corresponding unit fraction, are just like whole numbers, at least in terms of addition and subtraction.
(4) Students should get to know what a fraction is and what it means to add, subtract, multiply, and divide fractions before they perform the formal procedures of fraction arithmetic.
(5) The *least common denominator* is not needed for adding fractions, and there is no compelling mathematical reason to insist that fractions be always reduced to lowest terms.
(6) Finite decimals are a special class of fractions.
(7) Everything we need to know about fractions, including multiplication and division, can be explained using the definition of a fraction as a point on the number line.

(These emphases were first put forth in [Wu1998], and can be found in complete detail in [Wu2002]; they are also present in [Jensen].)

A second major departure lies in the heavy emphasis placed on geometry in the middle school curriculum, especially on giving precise definitions for the concepts of congruence and similarity. According to TSM, congruence means *same size and same shape* and similarity means *same shape but not necessarily the same size*. As mathematics, this is unacceptable because "same size" and "same shape" are words that can mean different things to different people, whereas mathematics only deals with clear and unambiguous information. What these volumes promote is a different approach to the teaching of these concepts. Take congruence, for example. First make sure that you know what translations, reflections, and rotations are, then devise hands-on activities for your students to familiarize themselves with these transformations, and, finally, teach them that, by definition, *two geometric figures are congruent if one figure can be carried onto the other by the use of a finite number of translations, reflections, and rotations.* Conceptually the same thing can be said about similarity. These volumes will help you acquire the requisite knowledge you need to teach congruence and similarity differently—and better.

The heavy emphasis on geometry all through both volumes is motivated by the fact that—contrary to what TSM would have you believe—familiarity with similar triangles is absolutely crucial to the learning of linear equations in algebra, particularly the concept of the **slope** of a line (see Chapter 3 of [NMP2], [Wu2010b], and [Wu2010c]). Students' understanding of the concept of slope is a main stumbling block in beginning algebra (see, e.g., [Postelnicu]), and one of the contributions of these volumes is a different approach to the definition of slope that is more intuitive and makes entirely obvious why certain lines have negative slope (see Section 4.3 in [Wu-Alg]).

While the geometric topics taken up are, with but one exception,[2] what one normally finds in the standard middle school curriculum—translations, reflections, rotations, congruence, length, area, volume, etc.—they are not taken up as fun, optional activities. Rather, these are topics that are essential for the learning of algebra and, to that end, are put to use in [Wu-Alg] for substantive logical reasoning in the discussion of the graphs of linear equations, linear functions, linear inequalities, and quadratic functions. For example, having a correct definition of the slope of a line makes it possible for teachers to *explain*, and for students to *understand* (rather than merely memorize), why the graph of a linear equation $ax + by = c$ is a line (see Section 4.4 in [Wu-Alg]). The absence of this reasoning in TSM has made the writing down of the equation of a line that satisfies certain geometric data a fearsome task to many students of algebra. But teachers who have been exposed to this reasoning will begin to see how they might teach the graphing of linear equations differently and liberate their students from this fear, because reasoning can now replace rote memorization.

Beyond the implications for the teaching of algebra, the other reason for the emphasis on geometry in the middle school curriculum is that translations, rotations, reflections, and dilations provide a much more accessible introduction to the staple of a rigorous high school course on geometry: the study of triangles and circles (cf. Volumes I and II of [Wu-HighSchool]). Because the learning of these transformations can be made more accessible and greatly expedited through the use of hands-on geometric experiments, the hands-on experiences serve to demystify congruence and similarity for students. At a time when the school geometry curriculum is beset by issues of fragmentation (because of the disconnect between middle school geometry and high school geometry) and meaningless abstraction (as a result of the rote application of the axiomatic method in a school setting), the middle course offered in these volumes is one potential solution to this pressing curricular problem. For a more detailed discussion of these ideas, see Section 4.1 on page 229.

[2]The one exception is the concept of *dilation*.

The final major departure from TSM in these volumes is the emphasis put on the careful use of symbols. The concept of a "variable" is at present the scourge of middle school mathematics that bars any meaningful entry into algebra. In mathematics, "variable" is no more than an *informal* piece of terminology that serves to remind us of an element in the domain of a function. Yet in TSM and the education literature, "variable" has been elevated to the status of a *mathematical* concept. The inevitable result of such an aberration is to make introductory algebra unlearnable. The whole of the companion volume [Wu-Alg] will testify to the fact that when careful attention is given to the correct use of symbols, rather than to the contortions involved in trying to make sense of "variable", every foundational concept and skill in introductory algebra (what is an equation? what does it mean to solve an equation? what is an expression? etc.) gains in clarity and conceptual simplicity, and algebra becomes once again a potentially learnable subject.

Although these two volumes (an expansion of [Wu2010b] and [Wu2010c]) have been used in my professional development institutes since 2006, it has been difficult to convince teachers to put such a mathematical development directly to use in their classrooms. Their reluctance is entirely understandable because doing so would entail the need to develop new classroom lessons—and probably new curricular units—on their own. It would also require them to teach *against* the existing curriculum of TSM. For example, according to TSM, fractions are best understood through the use of analogies and metaphors (compare the critique in pages 34–39 of [Wu2008]), the concept of a "variable" is central to middle school mathematics (page 102 of [NCTM]), and similar triangles are irrelevant to the learning of school algebra (look up almost any school algebra textbook in K–12 in the past four decades). It is unfair to ask teachers to single-handedly defy such an entrenched tradition.

This situation has changed somewhat with the advent of the Common Core State Standards for Mathematics (CCSSM) (see [CCSSM]). The CCSSM have come to substantial agreement with the main advocacies of these volumes,[3] especially the three major departures from TSM mentioned above. A recent article in *Education Week* ([Heitin]) indicates that, perhaps, educators have finally come around to embracing the main emphases on the teaching of fractions in (1)–(7) above (one can gain a little historical perspective on this issue by reading Chapter 24 of [Wu2011a]). It should now be easier to convince teachers to learn and apply the content of these volumes (and to convince their administrators to allow them to do so) because the CCSSM are now being implemented in most states. This fact acquires additional significance because on the one hand, school textbooks in general have not risen to the challenge of the CCSSM as of

[3]The document [Wu2010b] is the same document as the one cited as *"Wu, H. 'Lecture Notes for the* 2009 *Pre-Algebra Institute,' September* 15, 2009*"* on page 92 of [CCSSM].

November 2015, and on the other, there seems to be no other complete *mathematical* exposition of middle school mathematics that is consistent with the CCSSM—this is especially true for fractions, negative numbers, and geometry. My hope is that these volumes can double as a stopgap measure at a time when the implementation of the CCSSM seems not too sure of its mathematical footing.

An original impetus for the writing of these volumes was to help solve our nation's severe mathematics education crisis.[4] Back in 2004 when this work was first conceived, the CCSSM did not exist, but the glaring defects of TSM could not be ignored. There are good reasons to believe that the writing of the CCSSM was inspired by this same crisis. It is finally time to banish from schools the jumbled, chaotic, and even downright anti-mathematical presentations that characterize and pervade TSM. To this end, the present volumes strive to improve mathematics teaching by emphasizing, throughout, the following five *fundamental principles* (compare [Wu2011b]):

(I) *Precise definitions are essential.* In mathematics, precise definitions are the bedrock on which all logical reasoning rests, because mathematics does not deal with vaguely conceived notions. Yet definitions are looked upon with something close to disdain by most teachers (and students) as just "more things to memorize". Such a fundamental misconception of the basic structure of mathematics can only come from the TSM we all remember from our own schooling and now teach again to our students, and from the flawed professional development we provide for our teachers. In these volumes, we will respect this fundamental characteristic of mathematics by offering—and employing—precise definitions for every concept, including those that are commonly used, yet remain undefined, in TSM: *fractions, decimals, sum of fractions, product of fractions, division of fractions, ratio, percent, rate, equation, congruence, similarity, slope of a line, graph of an inequality, polygon, length, area,* etc.

(II) *Every statement must be supported by reasoning.* There are no unexplained assertions in these volumes.[5] If something is true, a logical explanation will be given. Although it takes some effort to learn the logical language used in mathematical reasoning, in the long run the presence of reasoning in all we do has the advantage of disarming disbelief and removing the stress of learning-by-rote. It also has the salutary effect of putting the learner and the teacher on the same footing, because the ultimate arbiter of truth will no longer be the teacher's or the textbook's authority, but the compelling rigor of the reasoning.

[4] See, for example, [Askey], [RAGS], and [NMP1].

[5] Except those few explicitly designated as such, because their proofs require advanced mathematics.

(III) *Mathematical statements are precise.* In mathematics, there is no room for imprecision because imprecision leads to misunderstanding and therefore errors. TSM, however, is rife with imprecision, saying things such as "the pizza is the whole" in the study of fractions. This leads to misconceptions about the "whole" being a *shape* (the circle), whereas what is meant mathematically is that the whole is the *area* of the pizza. TSM also defines *percent* to be *out of a hundred.* This then leaves students confused as to whether *percent* is an "action" or a number. If it is an "action", how does one add and divide "actions", and if it is a number, what kind of number is it? It is difficult to imagine how mathematics learning can take place when learners' minds are beset by such confusion. Another example is TSM's claim that "multiplication and division are inverse operations; they undo each other". But given two numbers such as 2 and 3, we have $2 \times 3 = 6$ and $2 \div 3 = \frac{2}{3}$. TSM does not explain in which way 6 and $\frac{2}{3}$ undo each other. What is meant is that if we fix a number k ($\neq 0$), then the operation of multiplying a given number by k followed by the operation of dividing the resulting number by k leaves the given number unchanged; in this sense, multiplication and division indeed undo each other. It would seem, however, that even this much precision is unattainable by TSM. This is another reason why TSM is unlearnable.

This lack of precision is by no means limited to elementary school mathematics; it pervades the K–12 curriculum. On the high school level, for example, the definition that $3^{-x} = 1/3^x$ is too often offered amidst a flurry of heuristic arguments that leave the readers with the impression that the equality $3^{-x} = 1/3^x$ has been *proved*. Such persistent ambiguities consequently leave many students as well as teachers confused about the difference between a definition and a theorem.

(IV) *Mathematics is coherent.* The concept of *mathematical coherence* is often brought up in educational discussions nowadays, but it is not something that can be understood through verbal descriptions any more than the transcendental serenity of the adagio in the Schubert C major quintet can be appreciated through the reading of an essay praising its beauty. Very crudely speaking, the coherence of mathematics refers to the fact that the body of knowledge that is mathematics has a tightly-knit structure, but the only way one can get to know and appreciate this structure is by wading into its details. For example, the concept of similarity in Section 4.7 on page 320 relies on a knowledge of multiplying and dividing fractions (Sections 1.5 and 1.6 on pages 56 and 70, respectively) and congruence (Section 4.5 on page 287), and is itself used in a crucial way for the definition of slope (Section 4.3 in [Wu-Alg]). Another example is the omnipresence of the theorem on equivalent fractions in the discussion of almost every topic in fractions, when TSM would have you believe that it is only useful for simplifying fractions. Yet another example is the key role played by congruence not only in the definition of similarity (Section 4.7 on page 320) but also in the considerations of length, area, and volume (see Chapter 5). As a final example, you will notice that the division of whole numbers, the division of fractions (Section 1.6 on page 70), and the division of rational numbers (Section 2.5 on page 174) are conceptually identical.

The coherence of mathematics makes mathematics more teachable and more learnable. This can be easily understood by an analogy: whereas one can pore over a page from a phone book without any recollection of what has been read afterwards, almost all readers have vivid memories of *Don Quixote*—all one thousand pages of it—even after only one reading, because it tells a coherent story.

Although coherence is difficult to describe, the *lack of coherence* can be more easily illustrated. A striking example of the failure of coherence in TSM is the common explanation of the theorem on equivalent fractions, which states that $\frac{m}{n} = \frac{km}{kn}$ *for all fractions* $\frac{m}{n}$ *and for all positive integers k.* TSM would have you believe that this is true because

$$\frac{m}{n} \;=\; 1 \times \frac{m}{n} \;=\; \frac{k}{k} \times \frac{m}{n} \;=\; \frac{km}{kn}.$$

However, the last step depends on knowing how to multiply fractions, and the multiplication of fractions is a topic that comes late in the development of the subject.[6] When the reasoning for *the* basic theorem in fractions—the theorem on equivalent fractions—is given in terms of something more complex and, in any case, not yet available, how can we expect students to learn? Unfortunately, such subversions of logic abound in TSM.

(V) *Mathematics is purposeful.* Mathematics is goal-oriented, and every concept or skill in the standard curriculum must be there for a purpose. Teachers who recognize the purposefulness of mathematics gain an extra tool for making their lessons more compelling and, therefore, more learnable. When congruence and similarity are taught with no apparent purpose except to do "fun activities",[7] students lose sight of the mathematics and may wonder why they are required to learn it. However, as noted above, the concept of congruence lies behind the concept of similarity, and both are needed to make sense of basic issues in algebra such as linear equations of two variables and their graphs, e.g., why is the graph of such an equation a (straight) line? Students are more likely to feel motivated to learn if presented with a curriculum that actually offers explanations of why its basic facts are worth learning.

[6] Multiplication is the most subtle among the four arithmetic operations on fractions. Its definition is nontrivial; the proof of the product formula is sophisticated; and its relationship with the area of a rectangle (with fractional sides) is subtle. See Section 1.5 on page 56.

[7] This has been happening all too often lately as a result of the misunderstanding of the CCSSM propagated by people immersed in TSM.

Middle school mathematics is the bridge that leads from fairly concrete concepts about numbers in elementary school to more abstract concepts in algebra, geometry, and trigonometry in high school. Our nation's curriculum is traditionally weak in middle school; one can almost say that it has been delinquent in its failure to provide careful guidance for students' transition from the concrete to the abstract. The teaching of TSM has become the norm in those years. Instead of giving precise instruction on the correct use of symbols and explaining the need for the idea of generality in students' next step on their mathematical journey, TSM harps on the alleged profundity of the fictitious concept of a "variable"; instead of guiding students' tentative first steps to think abstractly about negative numbers, TSM redirects them to *replace* abstract thinking by analogies and heuristic patterns. By contrast, this volume and its companion volume [Wu-Alg] take this bridge seriously. They confront the necessary abstractions without compromise, but they do so by building on the foundation of elementary school mathematics (cf. [Wu2011a]). I hope these volumes will initiate change by making you more aware of the overriding importance of this bridge in students' mathematics learning trajectory. Ultimately, the goal of these volumes is to help you teach your students better.

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