Preface

A main obstacle in the learning of school mathematics has always been how to cope with the steady increase in abstraction with the passage of each school year.

This volume and its companion volume—*Teaching School Mathematics: Pre-Algebra* ([Wu-PreAlg])—are textbooks written for teachers, especially middle school teachers. They address the mathematics that is generally taught in grades 6–8. In this volume, we give a presentation of school algebra as *a direct continuation of arithmetic*—whole numbers, fractions, decimals, and negative numbers—and we also assume a basic acquaintance with the geometry of congruence and similarity. For this reason, we must draw on the readers' knowledge of these topics. In the Appendix (pages 265 ff.), one can find a brief summary of most of the relevant facts from [Wu-PreAlg] that we need.

The topics to be taken up in this volume are those to be found in any middle school or high school course on Algebra I: linear equations in one and two variables, linear inequalities in one and two variables, simultaneous linear equations, the concept of a function, polynomial functions and exponential functions, and a detailed study of linear and quadratic functions. These topics are entirely unexceptional. Such being the case, one may well ask why this volume deserved to be written. In general terms, an answer to this question has been given in the Preface to [Wu-PreAlg]. What follows is a more focused answer in the context of the teaching and learning of introductory school algebra.

At the moment, *Algebra for All* is a national goal (see Chapter 3 of [NMP]), and there are various theories as to why this goal seems to be out of reach. Could it be that the appropriate classroom manipulatives have not been sufficiently exploited, that the latest advances in technology have not yet been fully integrated into the instruction, or that the teaching has slighted so-called sense-making, conceptual understanding, and higher-order thinking skills? Perhaps. All these questions, however, ignore a fundamental issue: there is ample evidence that students cannot learn algebra, not because they don't like the packaging of the product, but because they find the product itself to be incomprehensible. We will refer to this product—the mathematics in almost all the standard school textbooks of the past four decades—as **Textbook School Mathematics** ([**TSM**]).[1] TSM fails, often in spectacular fashion, to explain to students, *clearly* and *correctly*, what they are

[1]See, for example, [Wu2013] or [Wu2015] for more details.

supposed to learn. Education researchers who look into the nonlearning of algebra do not appear to have given much thought to the fact that the TSM that resides in student textbooks or standard professional development materials is riddled with ambiguities and errors, big and small. In short, TSM is not learnable. Until a mathematically correct version of school algebra is readily accessible to one and all, it will be premature to draw any conclusions about why students cannot learn algebra. With this in mind, the main justification for this volume's existence is that it gives a logical and coherent exposition of the standard mathematical topics in Algebra I in a way that not only is grade-level appropriate for eighth and ninth graders, but also meets the requirements of the following five *fundamental principles of mathematics*:

(I) Precise definitions are essential.
(II) Every statement must be supported by mathematical reasoning.
(III) Mathematical statements are precise.
(IV) Mathematics is coherent.
(V) Mathematics is purposeful.

We will refer the readers to the Preface of [Wu-PreAlg] for a fuller discussions of these fundamental principles.

The grade-level requirements we have imposed on this volume by no means imply that this is a student textbook. This volume is unequivocally a book for teachers with a sharp focus on mathematics. What this requirement means is that a conscientious attempt has been made to minimize the distance between the content in this volume and what teachers have to teach in middle school (see, for example, [Wu2006]). Consequently, this volume will not touch on any advanced topics such as vector spaces and linear transformations, groups, rings, fields, and especially finite fields. It turns out that the need for such advanced considerations is not critical at this stage and, in any case, there will be no advanced topics to be found in this volume. Instead, we will focus on probing the basic structure that undergirds the standard topics of school algebra. In the course of this probe, however, the need for advanced—and often quite subtle—considerations does surface from time to time. On these occasions, we will not shy away from giving the full explanation in order to bring mathematical closure to the discussion. All the same, we will also be explicit in pointing out that these advanced considerations are more for broadening the teachers' knowledge base than for school classroom presentations.

The fundamental principles of mathematics are of critical importance in the teaching of school algebra because algebra is inherently an abstract subject compared to arithmetic, and TSM's lack of precise definitions and logical reasoning in an abstract environment has rendered the subject unlearnable. In greater detail, let us consider the following specific manifestations of these flaws in the algebra portion of TSM:

1. TSM considers the concept of a "variable" to be basic in school algebra. For example:

> Understanding the concept of *variable* is crucial to the study of algebra; a major problem in students' efforts to understand and do algebra results from their narrow interpretation of the term. ([NCTM], page 102)

Many in the education establishment may be surprised to learn that "variable" is not a mathematically well-defined concept and is only used *informally* in mathematical discussions in order to remove excessive verbiage.[2] One should not expend scarce instructional time trying to teach a phantom concept, much less make it the cornerstone of algebra learning. When textbooks follow suit and elaborate on a "variable" as a quantity that changes or varies, they block beginners at the gate of the gate-keeper course that is algebra.

2. Once the concept of "variable" has taken root, an *equation* will naturally be defined in terms of a "variable". Here is a typical example:

> A *variable* is a symbol used to represent one or more numbers. A *variable expression* is an expression that contains a variable. . . . An *equation* is a statement formed by placing an equal sign between two numerical or variable expressions. ([Dolciani], pages 724 and 731)

This then raises the question of what it means for two variable expressions to be **equal**: if a variable can represent more than one number, does the equality of two variable expressions mean the expressions are equal for *all* the numbers so represented? If so, isn't that an *identity*? If not, then for *which* numbers are they equal?

When basic questions like these cannot be answered, it is a foregone conclusion that the fundamental process of *solving an equation*, in the way it is taught in school algebra, becomes a faith-based ritual divorced from mathematical reasoning (see the discussion on pages 37 ff.).

3. TSM introduces students to the concept of the *slope* of a nonvertical line strictly as a rote skill: fix two chosen points on the line and compute their "rise over run". There is no mention of the fact that, if two other points are chosen, the resulting "rise over run" will still be the same. Some students even ignore the "rise over run" and simply expect every line to come equipped with an equation $y = mx + b$ so that they can conveniently identify the slope of the line with the constant "m". Recently, the scope of the misconception about slope has been captured quantitatively in [Postelnicu-Greenes], but the education research literature still seems oblivious to the fundamental mathematical error in TSM's definition of slope and the glaring absence of reasoning surrounding this concept. Education research also appears to be unaware that, until this error is honestly confronted, it will be premature—not to say futile—to talk about students' "conceptual understanding" of slope.

4. A natural consequence of not having a correct definition of slope is the absence of any explanations for the interplay between a linear equation in two variables and its graph. For example, why is the graph of a linear equation in two variables a straight line? And is every straight line necessarily the graph of *some* linear equation in two variables? TSM's answer to the first question is that when several points in the graph of the linear equation are plotted, "they look straight". Reasoning plays no role. Consequently, students can only learn how to find the equation of a line satisfying certain geometric conditions (e.g., passing through two given points, passing through a given point with a given slope, etc.) as a rote skill. Since linear equations constitute a major part of the first half of Algebra I, this means that students' first encounter with algebra will

[2] We have already done so above by referring to "linear equations of one and two variables", etc.

consist mainly of a deeper immersion in learning-by-rote. After years of bruising battles with fraction-as-a-piece-of-pizza, students become convinced by such an encounter that math is unlearnable except by brute force memorization.

5. The theorem that two lines being parallel is equivalent to the lines having the same slope is routinely offered in textbooks as a *definition* or as a *key concept* of parallel lines. Likewise, the theorem that two lines being perpendicular is equivalent to the product of the slopes of the lines being equal to -1 is often given as a seemingly sophisticated *definition* of perpendicularity. Because students are already familiar with the concepts of parallel and perpendicular lines from earlier grades, they are confused by this spectacular about-face. Does a mathematical concept have any permanence, or is it liable to change with each grade? The likely conclusion from such confusion is that *algebra doesn't make sense.* This is one reason that the current discussion about "sense-making" in mathematics learning has no real traction: until we have a curriculum that makes sense, we cannot ask students to make sense of the mathematics.

6. In elementary and middle school, students have already used the concept of *constant rate* (e.g., *constant speed*) extensively, but there is no precise definition of this concept in TSM. What there is in TSM is an abstruse discussion of a concept called *proportional reasoning*; the implicit assumption is that if students have a conceptual understanding of *proportional reasoning,* they will be able to handle constant rate. An introductory algebra course is the first opportunity to bring clarity and closure to "constant rate" by pointing out what it means and why it corresponds to the linearity of an appropriate function. Yet this is hardly ever done. This is a prime example of the fractured school curriculum: the intrinsic coherence between the mathematics of grades 5–7 and the foundations of algebra is too often missing.

7. The concept of the *graph of an equation* is not precisely defined in TSM, and consequently not emphasized. It follows that simple facts about graphs such as the solution of simultaneous linear equations being the coordinates of the point of intersection of the two linear graphs become articles of faith rather than simple logical consequences of the definitions. Students do not learn *mathematics* if all they do is memorize facts on faith alone. Not surprisingly, some students do lose faith, which then makes any kind of learning—by rote or otherwise—impossible.

8. In TSM, the *graph of a linear inequality of two variables* is almost never defined, and the concept of a half-plane is also left undefined. Consequently, the theorem that the graph of a linear inequality is a half-plane becomes either a decree or a definition, and it is impossible to decide which it is. In asking students to learn about linear inequalities and linear programming, we are in effect asking them (once again) to wade through, and memorize by rote, a morass of disconnected shadowy statements while making believe that we are teaching mathematics. Under these circumstances, how can any mathematics learning take place?

9. The concept of a *rational*[3] exponent of a positive number is a source of immense confusion. TSM makes believe that, for any positive number a, $a^0 = 1$ is a theorem rather than a definition, and the same goes for $a^{-n} = \dfrac{1}{a^n}$ (for any positive integer n). Moreover, TSM does not explain that the reason we want a

[3]We are using the term of "rational numbers" in its correct mathematical sense: fractions *and* negative fractions.

definition of a^r for all rational numbers r is that these are special values of the exponential function $x \mapsto a^x$ when x is an arbitrary number. As a consequence, the laws of exponents become just another set of senseless rote skills about a strange notation rather than remarkable properties of the exponential function.

10. TSM's presentation of quadratic equations and functions is chaotic: too many facts to memorize while no conceptual framework is provided for their understanding. For example, students learn how to factor quadratic polynomials with leading coefficients other than 1, learn the quadratic formula, learn the formula for the axis of symmetry of the graph, learn the formula for the vertex of the graph of a quadratic function, etc. How are these related to each other?

If one goes through the algebra curriculum of TSM carefully, one will uncover these and many more serious mathematical issues. (Many of them will be pointed out in this volume in due course.) The prospect of a student learning algebra is therefore daunting: it may be likened to walking through a minefield where all the mines were put there by human errors. The least we can do is to remove the mines (and some of students' concomitant fears)—in other words, eradicate TSM—in order to give learning a chance. The modest goal of this volume is to give you the tools to do exactly that. Briefly, one will find in the following pages ways to deal with the preceding difficulties:

1*a*. What students should be learning is not what a "variable" is but the proper use of symbols; see pages 4 ff. The meaning of each symbol must be specified before it is put to use. For example, the equality of two functions of one variable, $f(x)$ and $g(x)$, may be a prototypical statement involving variables, but the precise *definition* of the equality $f = g$ is that, for each *fixed* number x in their common domain of definition, $f(x) = g(x)$. Nothing varies.

2*a*. The solving of equations is strictly a matter of computations with *numbers*. No variables are involved, and therefore there is no reason to confuse the issue by using balance scales or algebra tiles to explain the solution process. See the discussion in Section 3.1 on page 37.

3*a*. The concept of *slope* needs to be defined with far greater care than TSM has let on. One has to explain what "slope" tries to measure, how to measure it, and, most importantly, *why this way of measuring it is correct and useful.* In Section 4.3 on page 61, there is an extended discussion to this effect. In particular, this is where the discussion of congruent triangles and similar triangles in Chapter 4 of [Wu-PreAlg] becomes absolutely essential.

4*a*. In Sections 4.4 and 4.5 on pages 72 and 76, we will give a careful proof of why the graph of a linear equation of two variables is a line and why each line is the graph of *some* linear equation of two variables. In the process, it will become obvious how to write down the equation of a line that satisfies any of the standard geometric conditions. See Section 4.6 on page 78.

5*a*. Because perpendicularity and parallelism have been defined in Chapter 4 of [Wu-PreAlg], and because slope has been defined in Section 4.3 on page 61, any assertion about parallelism (or perpendicularity) and slope becomes a theorem to be proved. We will do exactly that in Sections 5.3 and 5.6 on pages 93 and 109, respectively.

6*a*. In Section 7.1 on page 137, we review the definition of constant rate, and then prove that constant rate is equivalent to the existence of an appropriate linear function that represents work done over time. In Section 7.2, we closely examine the possible meanings of *proportional reasoning* and point out how—by eliminating it altogether—its purported applications in school mathematics can all be put on a firm mathematical foundation.

7*a*. In Section 5.1, we explain precisely why the solutions to a pair of equations are the set of all the points of intersection of the graphs of the two equations in question. Such an explanation is possible only because the *graph of an equation* has been precisely defined and put to use in reasoning.

8*a*. In Section 8.4, we define the *half-planes* of a line and the *graph of a linear inequality*. Then in Theorem 8.4 on page 172, we prove that the graph of a linear inequality is a half-plane of the graph of the associated linear equation.

9*a*. Section 9.2 re-orients the discussion of rational exponents by assuming the existence of exponential functions from the beginning. (This is analogous to the discussion of solving polynomial equations by assuming—at the outset—the Fundamental Theorem of Algebra. In school mathematics, sometimes a central theorem has to be taken on faith for pedagogical reasons.) Then we make use of the characteristic property of the exponential functions (i.e., $a^x \cdot a^y = a^{x+y}$) to *prove* that $a^0 = 1$ and $a^{-x} = 1/a^x$. This makes it possible for the following section (Section 9.3) to present complete proofs of the other laws of exponents for rational exponents.

10*a*. Chapter 10 begins with a general discussion of the shape of the graph of a quadratic function and then shows how the graph can provide a framework for the understanding of quadratic functions in the same way that straight lines provide a framework for the understanding of linear functions. The basic technique here is that of *completing the square*; it will be seen that this technique unifies the diverse skills related to quadratic functions.

It can be persuasively argued that any form of professional development for middle school teachers that makes any claim to legitimacy must make the needed corrections of these flagrant errors in TSM. The content of this volume—in its various incarnations—has been used for both inservice and preservice professional development since 2006. Nevertheless, I have come to realize that, as of the year 2015, this offering comes with some liabilities. While it provides an opportunity for teachers to learn correct school mathematics, perhaps for the first time, it also obligates them to put in a tremendous amount of work in order to teach this material in the school classroom. In addition, the amount of steely resolve that is needed to teach it without the support of a compatible student textbook and a school's or a district's pacing guide may well be beyond the normal call of duty. To give a somewhat extreme example, if a teacher teaches slope more or less according to Section 4.3 on page 61 (see 3*a* above), then inevitably he or she will have to steal many hours from other topics in order to introduce students to the basic facts about similar triangles.

The advent of the CCSSM ([CCSSM]) should mitigate some of the difficulties teachers have in teaching correct algebra. If they wish to implement the content of this volume in their own classrooms, they can do so now with the assurance that, in the Common Core era, much of what used to be outlandish in this volume is now becoming the accepted norm. I can only hope that, in the forthcoming years, better student textbooks will be written so that the CCSSM will finally bring about better student learning in school algebra.

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