Introduction

"The Berkeley Math Circle was really critical in my development. It was the best method available not only to get a flow of mathematical ideas and problems to think about each week but also to meet other interested students and professional mathematicians from all over the Bay Area. You get stimulation from exchanging ideas with other people that you don't get from reading books at home.

I can also testify to the usefulness of studying mathematics even for students who don't plan on doing it as a career. For someone who wants to go into, say, law, policy analysis, philosophy, economics, or computer science, the kind of logical, abstract thinking that mathematics develops is really the best preparation. I realize that the Circle is most interested in attracting students whose lifelong passion is for mathematics, but it also helps others along the way."

Gabriel Carroll, BMC alumnus
Perfect IMO '01 score
Four-time Putnam Fellow
Ph.D. student in economics, MIT

1. Top-Tier Math Circles[1]

This book is based on material from a dozen of the 320 sessions of the Berkeley Math Circle (BMC), held over the past 10 years. In recent discussions, parents have described BMC as a *top-tier math circle*, calling for the following two definitions.

1.1. Math circles are weekly math programs that attract middle and high school students to mathematics by exposing them to intriguing and intellectually stimulating topics, rarely encountered in classrooms. Math circles vary in their organization, styles of sessions, and goals. But they all have one thing in common: *to inspire in students an understanding of and a lifelong love for mathematics.*

[1] Excerpts from [**98**] by Marc Whitlow and Mike Breen (BMC Parents), Zvezdelina Stankova (BMC Director), and Tatiana Shubin (SJMC Director)

1.2. Top-tier math circles *prepare our best young minds for their future roles as mathematics leaders.* Sessions are taught by accomplished mathematicians and explore advanced mathematical areas. They provide an educational opportunity for top pre-college mathematics students, not offered in any other setting in the U.S. education system. In addition to learning advanced mathematics topics, students are taught the technical writing skills needed to convey the solutions of complex problems.

As an example of a top-tier math circle, the **Berkeley Math Circle** is fashioned after the leading models in Eastern Europe, where math circles originated over a century ago. A typical weekly session has 20 to 30 students and lasts 2 hours. Like top-tier universities, BMC

- challenges students with beautiful, difficult mathematical theories,
- introduces them to powerful problem solving techniques,
- constantly provokes deep thought, and
- inspires the creation of original ideas.

Topics covered at BMC include combinatorics, graph theory, linear algebra, geometric transformations, recursive sequences, series, set theory, group theory, number theory, elliptic curves, algebraic geometry, applications to computer science, natural sciences, economics, and many more.

Each topic is taught by an *expert in the field* who has the ability to challenge the students and support them as they attempt to meet these challenges. All problems require students to come up with *mathematical proofs.* Proofs put forward by the students are not always the most eloquent. Only an accomplished mathematician can understand where a student might be heading in his/her proof and offer assistance through this challenge.[2]

The sessions are fast paced and intellectually demanding. It is hard to convey just how advanced this subject matter is without actually attending a session, but comparable levels can be found in advanced undergraduate and beginning graduate courses.

The *Monthly Contests* (MC)[3] at BMC can also convey the depth of the material. These are take-home exams of five hard, thought-provoking problems, requiring independent research. They develop not only advanced understanding, but also technical writing skills: the students must describe on paper, convincingly and without gaps, how they solved a problem. This is a fundamental skill and key to making intellectual property contributions; it is a unique feature of the top-tier math circles, not found in middle or high schools, where students are taught to meet state standards on questions that take less than a minute to answer. In contrast, monthly contest problems may take the best students hours or days of concentrated thought. Only a few participants are capable of solving all the problems; yet, through the attempt everyone learns about *the real world of mathematical research.*

[2] For examples of noteworthy past and present instructors who have brought their world expertise to BMC, see the Epilogue.

[3] Session 11 on *circle geometry* is based on the monthly contests.

1.3. The next generation of math leaders. The students of BMC come from a variety of socio-economic and ethnic backgrounds. The proportion of female to male students is approximately 2:3. This is an amazingly high fraction considering the trend of other high-level math programs, which are "male-dominated" or "male-only". Excellent role models for the female students are provided by the three female directors of the top-tier math circles in Berkeley [**8**], San Jose [**80**], and Los Angeles [**57**]; but perhaps even more important to the students are the outstanding lectures given by nine *female* professors and graduate students.

Currently, BMC does not actively recruit participants. Students and their parents find out about the circles in a variety of ways: from the circle's web site, `http://mathcircle.berkeley.edu/`, by word of mouth, through a local university, and in publications. There is no formal selection process, and *the math circle doors are open to any student.*

Needless to say, the BMC students are usually years ahead of their peers: they often complete most of high school mathematics by age 13 (8^{th} grade), some take many college math major courses by the time they graduate from high school, and a few of the top circlers venture into graduate courses and serious mathematical research even before entering a university. The *accomplishments of students* who have benefited from BMC can be measured in many ways. For example, a number of these students have gone on to win International Math Olympiad medals and Putnam awards, and the majority have been admitted to top-tier universities. BMC and the other top-tier math circles not only produce highly accomplished students – they produce and train *the next generation of leaders in mathematics.*[4]

2. Why, What, and for Whom?

Running the Berkeley Math Circle for ten years has taught us many lessons about math education in the U.S. and has even helped us understand better our own childhood education and origins of our passion for mathematics. To share this experience with you, the reader, is the *purpose of this book*:

- to present you with beautiful theories, problem solving techniques, and mathematical insights;
- to provide you with an abundance of exercises and problems to work on and with ready materials for math circle sessions.

2.1. The middle or high school student who is interested in expanding his/her math horizon and going well beyond anything that the regular math classroom can offer, who is brave enough to tackle non-trivial math ideas and work on hard problems for hours, who loves challenges and is motivated to overcome them: this is *the ideal reader of the book.*

[4] To learn about the need for top-tier math circles, we direct the reader to the Epilogue.

Don't confuse the above description with "top" or "brilliant" student: you will never know if you are talented in math unless you give it a try. And you may be pleasantly surprised by what you find out: that mathematics is a whole lot more than "adding fractions," "algebraic manipulations," or "endless quadratic equations" in homework assignments. You will discover that calculus is not the "pinnacle" of mathematical knowledge (as thought by the general public): it is only one of many beginnings, part of the subject of real analysis. Indeed, *other* wonderful topics are awaiting you here:

- inversion in geometry and circle geometry,
- abstract algebra via Rubik's cube and number theory,
- a mass point "hybrid" between algebra and geometry,
- combinatorics and complex numbers,
- game theory via invariants and monovariants, and, of course,
- plenty of proof methods and problem solving techniques.

2.2. Prerequisites. To read the book comfortably, you do *not* need to have calculus under your belt. However, familiarity with basic geometry and algebra concepts and theorems will definitely be helpful, e.g., lines, circles, triangles, rectangles, trapezoids, and quadrilaterals in general; similarity criteria for triangles and the Pythagorean Theorem; equal alternate interior angles for parallel lines and bisecting diagonals in a parallelogram; integers, divisibility and remainders; operations on fractions and real numbers, intervals and sets of numbers; and manipulations of algebraic expressions written with letters. Occasionally, functions will play an important role; hence having studied some basic (pre-calculus) examples will not hurt, e.g., linear and quadratic functions, polynomials, exponential and trigonometric functions, as well as their graphs.

The above concepts will be re-introduced via examples in the book. But if you feel that you need more solid background, we direct you to several wonderful books that should be part of any budding mathematician's library:

- *Geometry, Book 1* by Kiselev (cf. [**51**]),
- *Algebra* by Gelfand and Shen (cf. [**35**]),
- *Functions and Graphs*, *The Method of Coordinates*, and *Trigonometry* by Gelfand et al. (cf. [**33, 32, 34**]),
- for the older reader, *103 Trigonometry Problems from the Training of the USA IMO Team* by Andreescu and Feng (cf. [**3**]).

2.3. The middle or high school teacher who wishes to start a math circle in his/her school or teach a specially designed problem solving class will find this book invaluable. Most of the topics are *introductory* and independent of each other, and hence they can be pursued in just about any order during the first year of a new math circle. Yet, as you read through the sessions, you will notice how some of them also contain harder material suitable for *intermediate* level and the second year of a math circle. Other than the several open problems sprinkled throughout the text, the only truly *advanced* part is the *Mansion Appendix* in the last session: it is original research by BMCer Evan O'Dorney and can be left for the die-hards.

Running a math circle, especially for a teacher, is a hard task. But it is possible. In the 1960's, Tom Rike (an editor for this book and a veteran high school math teacher) was working on his master's degree. While browsing in the library one day, he ran across *The USSR Olympiad Problem Book* (cf. [**82**]). It contained problems written for talented 7^{th}–10^{th} graders; yet, he could not solve any of these "elementary" problems. In his own words:

> *"My abstract algebra had been too* abstract*, and I did not have the concrete examples that I needed. I never took a class in number theory because it sounded too elementary. I had developed the real number system starting from the Peano axioms, but I didn't really understand the fundamentals of the natural numbers, prime numbers. This was an epiphany for me. I felt as though I had been challenged by some force outside me and did not know how to respond."*

For the next 30 years Tom studied olympiad problem solving, first on his own, then through workshops and math circles in the SF Bay Area. He ran his own math circle at Oakland High School and gave talks at just about all other circles around. Even though at times he was only "a few pages" ahead of the students, he kept on learning and teaching problem solving because working on math circles had come to be a large part of his life:

> *"Although I have not attained my goal of becoming a true olympiad problem solver, the journey I have made in pursuit of this goal has been one of the most rewarding endeavors in my life."*

Hence, a word to the middle and high school teachers: keep on reading the book, despite moments of difficulty or confusion. For the motivated, persevering, and caring teacher, there will come a time when he/she will look back at the material here, smile, and effortlessly deliver it to the students at his/her own math circle. Truly gratifying.

2.4. Proofs in particular. The only topic that has been deliberately postponed till mid-book is *Proofs* in Sessions 5, 6, and 8. That proofs are important goes without question in Galileo's mind:

> *"It appears to me that those who rely simply on the weight of authority to prove any assertion, without searching out the arguments to support it, act absurdly. I wish to question freely and to answer freely without any sort of adulation. That well becomes any who are sincere in the search for truth."*

We shall learn a variety of proof methods: by contradiction, Pigeonhole Principle, and induction; by counterexample, example, or general argument; using invariants or monovariants, and others. The first session already calls for the method of inversion coupled with direct proofs, extra constructions, and contradiction.

The delay in formally introducing proofs is justified by our belief that humans and machines learn in essentially different ways. If we teach a machine how to do math problems, the only way for it to be perfectly "happy" is to receive the information in a *linear fashion*, i.e., to be fed first several chapters strictly on proofs and then to proceed with examples and more sophisticated problems from various math disciplines. On the other hand, human beings are curious and somewhat contradictory creatures: even though eventually we would like everything to be molded into an elegant, logically stable "tower" of knowledge, we also insist on seeing justification before we plunge into big enterprises. A young student, especially, needs *to know why a proof is important* and hence be motivated to search for proofs, instead of being turned off (as so often happens in proof-classes in school) by the stricture of long two-column proofs of relatively easy but "boring" statements.

And thus, we opted to expose the reader first to four wonderful and intellectually stimulating topics, inversion in the plane, combinatorics, Rubik's cube, and number theory. These sessions don't avoid proof techniques; yet, they mainly concentrate on a specific mathematical topic. Hopefully, this way the beginner will be drawn to the beauty of each subject, will accumulate interest, ideas, see examples, and be convinced that without proofs we can't really claim something is true. After such an initial level of mathematical maturity is acquired and the reader is "won over" to the math problem solving "front", we will formally welcome proofs in Session 5.

Teachers who plan on using the book in circles or classes should consider delivering the material in this order, postponing formal proofs until students have been exposed to at least two or three intriguing mathematical topics.

2.5. The parent of a middle or high school student is also among our intended audience; in fact, parents are probably *the most important readers* because without their support and enthusiasm, without them bringing and encouraging their children, there would hardly be any top-tier math circles in the U.S. Hence, if you are among those parents or if you are a parent new to the math circle movement, this book will provide a very strong beginning for your child. And for you as well.

The back row of the BMC classroom is always packed with adults: mostly parents and teachers. A strict rule reigns in this last row: you can never speak or interfere during sessions. Yet, as you can imagine, any rule is there to be broken: it is parents that become at times exulted by the material presented and cannot keep quiet, shouting involuntarily an answer to one or another (unintentionally) provocative question.

As a parent, you can do three things with this book: give it to your child (but make sure that he/she has the necessary background – see the recommended basic books); learn from it and teach your child; or give it to his/her math teacher and encourage the founding of a school-based math circle. Whatever path you choose to follow, it will eventually benefit your child and possibly a larger group of classmates. In any case, enjoy the book!

3. Notation and Technicalities

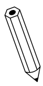
"Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures without which it is humanly impossible to understand a single word of it; without these one is wandering in a dark labyrinth."

Galileo

3.1. Marginalia. In addition to geometric "characters", we will also use a number of other symbols from algebra and logic. Let us examine first the non-standard margin icons which appear throughout the book.

 Warm-up or brute force

 Exercise

 Problem

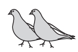
 Open question or one that requires extra knowledge

 Problem solving technique

 Warning

 Contradiction

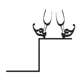
 Basic Pigeonhole Principle

 Generalized Pigeonhole Principle

 Basis Step

 Inductive Step

 Strong Basis Step

 Strong Inductive Step

The first four margin pictures refer to increasing difficulty of exercises and problems. Assigning such symbols is somewhat arbitrary since the same exercise could be *easy* for one person and could be a really *hard* problem for another; something may be beyond the knowledge of the reader *early* in the book, while *later* it may turn out to be a piece of cake. Thus, treat these symbols as a general guide to the difficulty of the material and make your own judgment after having attempted each problem.

The problem solving techniques, indicated by an eye, are ubiquitous throughout the book and will be discussed in the next section. The warning road sign, the high-voltage symbol, and the pigeons will appear prominently in Session 5 on proofs. The last four margin pictures refer to the steps of basic and strong *mathematical induction*, to be introduced in Session 6.

3.2. Logic. Mathematical statements that are proven are referred to by standard names such as *theorem, lemma, proposition, property,* or *corollary. Conjectures* are statements that are believed to be true, but no proof for them has been supplied yet. We will avoid the formal *definition* environment whenever possible, but theorems and such will be phrased formally.

Almost all sessions have a section on *Hints and Solutions to Selected Problems*. There and throughout the text, you will see two symbols indicating the end of a solution. The standard square $\square$ indicates the end of a *complete* solution or a proof with minor gaps, which are usually mentioned and the reader is expected to easily fill them in. The diamond $\Diamond$ is at the end of an *incomplete* solution, partial proof, sketch of a proof, hint, or any discussion requiring more work by the reader to reach a complete proof.

The text uses standard mathematical words and expressions, such as "implies," "therefore," "if then," "only if," "if and only if," letter notations for various sets of numbers, e.g., $\mathbb{Z}$ for the integers, and many others. Even though some are explained and illustrated via examples, the reader is expected to be familiar with basic logic notions and notation (cf. list of *Symbols and Notation* on page 299). If you need to review or learn this material in depth, we refer you to the first chapter of Jacobs' *Geometry* [**47**] on *deductive reasoning*. A complete list of *Abbreviations* can be found on page 301.

3.3. Labeling and future volumes. Subfigures within the same figure are implicitly labeled in alphabetical order. For example, Figure 7 on page 7 contains subfigures Figure 7a, 7b, and 7c, reading from left to right. Finally, more than half of the sessions are Parts I of series of sessions, to be continued in Volumes II and III of the book.

4. The Art of Being a Mathematician and Problem Solving

"Perhaps I can best describe my experience of doing mathematics in terms of a journey through a dark unexplored mansion. You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture, but gradually you learn where each piece of furniture is. Finally, after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated. You can see exactly where you were."

Sir Andrew John Wiles

There are no manuals on how to become a mathematician. This book will give you tips and point to possible paths, but the "art of being a mathematician" can be mastered only through personal experience. With every problem solved and every new definition or theorem learned, you will move closer to this goal. The two most important skills that you will acquire along the way are

- to think creatively while still "obeying the rules" and
- to make connections between problems, ideas, and even theories.

4.1. Problem solving techniques. Although all sessions in this book are based on basic knowledge from middle and high school and are, therefore, accessible to a wide range of ages and mathematical backgrounds, to do the exercises, you need to develop *problem solving techniques* (*PST's*). Session 1 on inversion will introduce PST's as part of a trilogy of mathematical knowledge: Concepts, Theorems, and PST's; and throughout the book you will encounter about 100 PST's. You will also need to learn how to *fit together various mathematical parts* in order to move forward in the solutions.

4.2. Muddying your hands. Do not expect each session to be a collection of clearly spelled out recipes leading to instantaneous solutions .... Nope! The book will encourage you to apply the newly acquired knowledge to problems and will guide you along the way but will rarely give you ready answers. *"The best way to learn is to learn from your own mistakes,"* said my advisor Joe Harris. Sessions 5, 6, and 8, for example, will formally discuss non-proofs, and a number of other sessions will present common problem solving pitfalls.

And so, it will be you, the reader, who has to commit to mastering the new math theories and techniques by

- "muddying your hands" in the problems,
- going back and reviewing necessary PST's and theory, and
- persistently moving forward in the book.

Nothing good comes "for free": you will have to work hard, always with a pencil and paper in hand. Keep in mind that the math world is huge: you'll never know everything, but you'll learn *where* to find things, how to connect and use them. The rewards will be substantial.

5. Acknowledgments

5.1. Institutional support and sponsors. The Berkeley Math Circle was made possible through the years with the unwavering support of:

• *University of California at Berkeley Math Department*, which hosts the Circle and its web site and has provided student assistants and secretarial support every year since 1999. Through faculty grants, Ivan Matić has been able to act as an associate director. The department chairs Cal Moore, Hugh Woodin, Ted Slaman, and Alan Weinstein have always been encouraging and supportive, and 16 UCB professors have delivered Circle sessions.

• *Mathematical Sciences Research Institute*, which has overseen the project from its inception, provided funds through various sponsors, and hosted Circle meetings and events. Special thanks to Deputy Directors Hugo Rossi, Joe Buhler, Michael Singer, and Bob Megginson, Directors David Eisenbud and Robert Bryant, and Associate Director Kathy O'Hara for their leadership, understanding, and help.

A number of *sponsors* have financially supported BMC over the years: Packard Foundation, Toyota Foundation, Clay Mathematics Institute, Mosse Foundation for Art and Education, Merriam-Webster Foundation for the

Scripps National Spelling Bee; National Science Foundation and other grants from Professors Ravi Vakil (Stanford), Bjorn Poonen, Alexander Givental, and Martin Olsson (UC Berkeley); and generous private donors.

5.2. Parents and students. *The BMC parents* have encouraged and driven their kids to the Circle for years, brought snacks during the breaks, organized Circle parties, attended meetings, and donated time, effort, and personal funds to the Circle. We are especially grateful to Marc Whitlow, Mike Breen, Jennifer O'Dorney, Yuki Ishikawa, and Tony DeRose for their enthusiasm, leadership, and professional services provided so selflessly to BMC.

A sequence of UC Berkeley *student assistants* have contributed to the smooth operation of the Circle by communicating with circlers, parents, instructors, and administrators and by re-designing and maintaining the web site. Joyce Yeung, Maksim Maydanskiy, Wycee de Vera, William Chen, and David Wertheimer have been exceptionally professional and caring. Many thanks go to our *monthly contest coordinators*: Professors Alexander Givental and Bjorn Poonen, circlers Gabriel Carroll, Andrew Dudzik, Inna Zakharevich, Neil Herriot, Maksim Maydanskiy, and Evan O'Dorney, and associate director Ivan Matić.

5.3. Professional support with the web site has been rendered on numerous occasions by Paulo de Souza, Dmitri Mironov, and Steve Sizemore. Marsha Snow, Barbara Peavy, and Tom Brown have offered valuable *secretarial support* over the years. BMC owes its *logo design* to Archer Design Inc.

As one can see, dozens of people have been involved in running the Berkeley Math Circle: it is a joint operation born of the love and care for our young generation of mathematicians. The most important people in this operation are undoubtedly the **BMC instructors** (over 50), who have delivered the 320 sessions during the last 10 years. We would like to thank all of them! Twelve joined BMC a decade ago and have stayed with us throughout the years: Ted Alper, Tom Davis, Dmitry Fuchs, Alexander Givental, Quan Lam, Bjorn Poonen, Tom Rike, Vera Serganova, Tatiana Shubin, Zvezdelina Stankova, Paul Zeitz, and Joshua Zucker.

5.4. Book support. Edward Dunne, our AMS editor, and his staff have been very helpful in resolving technical and other issues. Gabriel Carroll is responsible for drawing half a dozen cartoons, inspired by the earlier BMC sessions. All AHSME, AIME, and USAMO problems are used with permission from the American Mathematics Competitions (AMC), Lincoln, Nebraska [**1**]. A number of pictures and references have been taken from Wikipedia at `http://www.wikipedia.org/`.

With gratitude,
Zvezdelina Stankova
Berkeley Math Circle Director