Introduction

Our book, *Integers, Fractions and Arithmetic,* is a comprehensive and careful study of the fundamental topics of K - 8 arithmetic. This guide aims to help teachers understand the mathematical foundations of number theory in order to strengthen and enrich their mathematics classes. There are numerous activities that are suitable for teachers to bring into their classrooms.

The far-reaching point of view taken in the twelve seminars comprising this book is meant to enhance teachers' expertise. It is not meant for students learning the subject for the first time.

These seminars are designed for the professional development of teachers. They are also intended to be used in teacher preparation programs for undergraduates and graduates. They are appropriate for educators conducting enrichment programs such as Math Circles for Teachers, as well.

Conversations between the seminar leader and the other participants are an essential component of these seminars. Participants' input is important. There are many Seminar Exercises, Seminar Discussions and Seminar/Classroom Activities throughout this book. Comments on each of these are included as part of the main text.

Here is a brief outline of the topics and special features of each of the twelve seminars.

• Seminar 1, **Number Systems,** introduces the concept of a number system and its arithmetic operations. The natural numbers, the whole numbers and the integers, for example, are much more than just static collections of numbers. Each of these collections has a framework mandated by the properties of its operations of addition and multiplication.

• Seminar 2, **Divisibility and Order in the Integers,** introduces the idea of "dividing evenly" or divisibility, a basic and important topic in the theory of numbers. The properties of divisibility and its interactions with addition and multiplication are explained. The concept of divisor is defined. As an introductory illustration of the connection between divisibility and cryptography, we present a secret code game for the classroom. Order and its properties are discussed, as is the representation of integers by points on the number line.

• Seminar 3, **GCD's and The Division Algorithm,** examines the greatest common divisor (gcd) of a pair of integers. Several methods for calculating the greatest common divisor of two integers, including the Euclidean Algorithm, are explained and numerous examples are given. The division algorithm, or integer division with remainder, is introduced. It is a technique that is employed in every one of the succeeding seminars. The greatest common divisor of two integers a and b is shown to be a linear combination $sa + tb$. A feature of this seminar is an algorithm, the coefficient algorithm, that calculates the integers s and t.

• Seminar 4, **Prime Numbers and Factorization Into Primes,** begins with the definition of a prime number. Very quickly, using the properties of order, a simple useful tool for recognizing primes is presented, as is the Sieve of Eratosthenes, a method for finding all primes less than a fixed number. The sieve serves as an engaging activity for the middle grade classroom. Euclid's proof that there are infinitely many primes is discussed and the Fundamental Theorem of Arithmetic is explained. This theorem establishes the prime numbers as the "building blocks" of the integers.

• Seminar 5, **Applications of Prime Power Factorization,** explains how to use the prime power factorization of a positive integer $n > 1$, to count the number of divisors of n, and how to tabulate them. Of particular interest is the prime factorization of a perfect square. The information gleaned is used to study the "Locker Problem." The prime power factorization is used to calculate the greatest common divisor and the least common multiple of two positive integers, and the least common denominator of two positive fractions. The seminar concludes with more code games for students to show them how useful the understanding of primes and divisors can be.

• Seminar 6, **Modular Arithmetic With Applications to Divisibility Tests,** uses the examples of a light switch and a clock to introduce congruence and modular arithmetic. The concept of congruence is defined, and addition and multiplication of congruences (modular arithmetic) are explored. The properties of the operations of modular arithmetic are verified, and their consistency is discussed. The primary application of congruence in this seminar is the revelation that congruence is the basis of the popular classroom tests for divisibility.

• Seminar 7, **More Modular Arithmetic**, explores the notion of congruence classes for a particular modulus $m > 1$ and shows how the set of integers is partitioned into m nonintersecting congruence classes. The intriguing idea that addition and multiplication can be defined on the m congruence classes themselves leads to a number system, denoted $\mathbb{Z}_m$, with precisely m numbers. Linear congruences are defined and their solutions discussed.

• Seminar 8, **The Arithmetic of Fractions**, is the first of five seminars dedicated to fractions and decimals because of their importance in the classroom curriculum. The standard topics of addition, multiplication, division, common denominators and equivalent fractions are covered in detail in these five seminars. However, these topics are arranged in an order that is slightly different from the usual one. Multiplication is treated first, and with that in hand, common denominators and equivalent fractions are more readily understood and are available for use when discussing addition. Seminar 8 explains what a fraction is. The definition and properties of multiplication of fractions are developed. The concept of a common denominator of a pair of fractions, and two algorithms for finding the least common denominator are discussed. One uses the prime factoriztions of each of the two denominators. The other applies a formula involving only the product of the denominators and their greatest common divisor, allowing the calculation of the least common denominator by means of the Euclidean Algorithm. Equivalence of fractions is defined, its properties and many examples are studied. The cross product criterion for equivalence of two fractions is derived. The final topic in this seminar is equivalence classes of fractions and the fact that, in each equivalence class there is a fraction with smallest positive denominator. It is the fraction in "lowest terms."

• Seminar 9, **The Properties of Multiplication of Fractions**, begins by explaining the connection between a fraction in lowest terms and the more familiar concept of a fraction with numerator and denominator having no factors in common, other than ± 1. The important fact that multiplication respects equivalence is verified and the properties of multiplication are established. Other topics covered include mixed numbers and the division of fractions. The seminar end with a collection of "Word Problems." Four examples are given and three more are part of a Seminar/Classroom Activity. (Solutions are included as part of the text.)

• Seminar 10, **Addition of Fraction**, highlights the three step "Sure Fire Method" for finding the sum of two fractions. (The name derives from the fact that every problem on addition of fractions is solved the same way.) This method of addition is compared to the method using the least common denominator, and numerous examples are computed both ways. The properties of addition are verified. The additive inverse of a fraction is carefully explained and subtraction is defined in terms of the additive inverse. The essential distributive property that links multiplication and addition of fractions is discussed. The final topic establishes the consistency of addition of fractions with respect to equivalence.

• Seminar 11, **The Decimal Expansion of a Fraction,** defines the decimal expansion of a fraction a/b, with $0 < a < b$, as a sum of decimal fractions and describes how to construct the decimal expansion by means of repeated use of the division algorithm. (In Appendix A, the "long division algorithm" is derived from the same procedure.) Terminating and nonterminating decimals are defined. An important result is that a fraction a/b, in lowest terms, with $0 < a < b$, has a terminating decimal expansion precisely when its denominator b is a product of powers of 2 and/or 5. In the section on nonterminating decimals, the repetend of a/b, with $0 < a < b$, is defined as the sequence of place values of a/b that repeats. It is shown that the repetend begins at the 10^{-b}ths place or earlier. A highlight is the discussion of the fact that if b is relatively prime to 10, then the repetend begins in the tenths place, whereas if b is not relatively prime to 10 and has factors other than 2 or 5, then the repetend begins in the 10^{-j}ths place, where $j \geq 2$.

• Seminar 12, **Order and the Number Line,** defines order on the set of fractions as a natural extension of order on the integers, and verifies that equivalence respects order. The cross product comparison rule for fractions with positive denominators is derived. Comparison of fractions translates into a method for comparing decimals by comparing their place values. The facts that the set of positive fractions does not have a least element and that it is always possible to construct fractions between two given fractions are discussed. Next, attention is directed to the number line and the representation of fractions and decimals as points on this line. Order determines the position of these points on the number line. The distance between two nonequivalent fractions is defined, and it is seen that it is always positive. Furthermore, it is shown that the distance between two nonequivalent fractions can be estimated by comparing place values in their decimal expansions.

Acknowledgements. We acknowledge with gratitude the assistance of John Boller, Jonny Gleason, Nathan Hatch, Ryan Julian, Sara Mahoney, Xiao Xiao Peng, Weston Ungemach and Michael Wong. We thank the National Science Foundation for partial support, under Grant #NSF ESI-0101913, during the preparation of this manuscript.