Seminar 1

Number Systems

We set the conversational tone of these seminars by involving everyone in a mathematical discussion right from the start. We begin this first seminar with a Seminar/Classroom Activity. These activities are an integral part of each seminar and are suitable for use in the classroom.

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Seminar/Classroom Activity. For this activity, we focus on mathematical words related to numbers and arithmetic. The seminar leader will ask each of the participants, in turn, to write on the board a mathematical word associated with the topic Integers, Fractions and Arithmetic. Basic words, such as "sum" or "integer" are welcome. Complicated words are not so suitable here. Also, we suggest that the more variety of words put forward, the better and the more value for everyone. We hope mathematical conversations will take place as the words are put on the board.

After everyone has written a word on the board, the seminar leader will ask each participant to choose a word and use it in a sentence. For example, for the word "sum" we might say, "The sum of 3 and 5 is 8." For the word "integer" we might say, "-1 is a negative integer."

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Comments on the Seminar/Classroom Activity. (These comments are meant to be read after the activity is concluded.) Here are ten words associated to the topic Integers, Fractions and Arithmetic, and ten sentences using the words.

equation	negative
fraction	odd
exponent	subtract
less	factor
algorithm	prime

• The *equation* $2x = 3$ has the solution $x = 3/2$.
• A *negative* integer is represented by a point on the number line to the left of 0.

• The *fraction* 1/2 is not an integer.
• 1 is an *odd* integer.
• The integer 7^8 has *exponent* 8.
• *Subtract* the amount of the withdrawl from the balance in your account.
• The integer −1 is *less* than the integer 1.
• 5 is a *factor* of every integer with units digit equal to 0 or to 5.
• An *algorithm* is a procedure for solving a problem in a finite number of steps.
• The integers 2 and 3 are *prime* numbers.

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These words, and many others, will be discussed in the course of these seminars. It is interesting to observe that many if not most of the words listed above refer not just to an individual number but to connections between numbers. For example, to tell whether an integer is prime or not, we must look at the integers that are its factors.

In this seminar, we explain the concept of number system, and examine familiar sets of numbers and their underlying structures. The natural numbers, whole numbers and integers, for example, are much more than just static collections of numbers. Each of these collections has a framework mandated by the properties of its operations of addition and multiplication.

1. Arithmetic in the Integers, Part I

The integers are far more than just the elements 0, 1, −1, 2, −2, ... The collection of integers, which we denote by $\mathbb{Z}$, is a set endowed with two operations, addition and multiplication, that have very important properties. Adding and multiplying are processes, but it is the rules of addition and multiplication that form the essence of arithmetic. We will define the word "operation" formally in Section 4 where we work with you to deduce some very important facts from the rules which we list now.

We restrict our attention to the integers in this section. First, we describe the rules for addition, then the rules for multiplication, and, finally, the essential rule, known as the distributive property, that connects addition and multiplication. The first rule formalizes the idea that when we add two integers, we obtain an integer.

(A1) Closure. If a and b are in $\mathbb{Z}$, then $a + b$ is in $\mathbb{Z}$.

For example, if we add the integers 53 and 101, the sum is the integer 154. If we add the integers 2 and −3, the sum is the integer −1.

If we want to add the integers 4, 6 and 21, we can first compute the sum $4 + 6 = 10$, and then add 21 to obtain $10 + 21 = 31$. Or, we can add 4 to the sum $6 + 21 = 27$ of 6 and 21 to obtain $4 + 27 = 31$. In both cases, the sum is the same. The associative rule, which we define next, tells us that this is always the case. It states that no matter how we associate, or group, numbers for addition, the sum is the same.

(A2) Associative Rule. If a, b and c are in $\mathbb{Z}$, then

$$(a+b)+c = a+(b+c).$$

Not all familiar operations on $\mathbb{Z}$ are associative. For an example of an operation that is *not* associative, consider exponentiation.

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Seminar Exercise. Find three integers a, b and c such that

$$(a^b)^c \neq a^{(b^c)}.$$

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Comments on the Seminar Exercise. One such example is $a = 2$, $b = 2$ and $c = 3$. In this case, $(2^2)^3 = 64$, whereas $2^{(2^3)} = 2^8 = 256$. Consequently, when you teach exponentiation, you need to be prepared to discuss associativity.

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The next rule states that we can add integers in any order.

(A3) Commutative Rule. For integers a and b,

$$a+b = b+a.$$

For example, $22 + 31 = 53$ and $31 + 22 = 53$. Also, $(-4) + 9 = 5$ and $9 + (-4) = 5$.

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Seminar Question. Is exponentiation commutative? Give an example that justifies your answer.

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Comments on the Seminar Question. Exponentiation is not commutative as the example $2^3 = 8$ and $3^2 = 9$ shows.

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Note that the combination of Rules (A2) and (A3) gives us many more options for grouping and ordering the sum of three numbers. For an example, we return to the sum of the three numbers 4, 6 and 21. Rules (A2) and (A3) allow us to first add 4 and 21 and then add 6. Here is one of several arguments showing why this is so.

$$\begin{aligned}(4+6)+21 &= 4+(6+21), \quad \text{by (A2)}\\ &= 4+(21+6), \quad \text{by (A3)}\\ &= (4+21)+6, \quad \text{by (A2)}\end{aligned}$$

This is a nice exercise for students who are learning the associative and commutative rules.

Be aware. The final two rules for addition of integers, (A4) and (A5), do not hold in some of the number systems we work with. We will discuss this in Section 3.

The next rule states that in the integers there is a particular integer with a very special property for addition. This integer is a solution x to the equation

$$a + x = a,$$

for all a in $\mathbb{Z}$. Such an element x is called an *additive identity.* The rule identifies the additive identity in $\mathbb{Z}$, and states that it is unique.
(A4) Additive Identity. There is precisely one element in $\mathbb{Z}$, called zero and denoted 0, such that

$$a + 0 = 0 + a = a, \text{ for all } a \text{ in } \mathbb{Z}.$$

Uniqueness means that there is only one solution, namely, the one we call 0, to the equation $a + x = a$, *for all a in $\mathbb{Z}$*. Consequently, if there is another integer z that satisfies $a + z = a$, for all integers a, then $z = 0$.

Since $\mathbb{Z}$ has an additive identity, 0, it makes sense to ask if, for each integer a, there is a solution x to the equation

$$a + x = 0.$$

The last rule for addition in the integers states that, for *every* integer a, there is precisely one such solution. The solution x depends on a, and is called the *additive inverse of a.*
(A5) Additive Inverse. For every a in $\mathbb{Z}$, there is a unique element, denoted $-a$, in $\mathbb{Z}$ such that

$$a + (-a) = 0.$$

Note that by (A3) we also have

$$(-a) + a = 0.$$

Uniqueness of the additive inverse of an integer a in $\mathbb{Z}$, means that if we find an integer b such that $a + b = 0$, then $b = -a$.

We put this idea to use immediately to find the additive inverse $-(-a)$ of the integer $-a$. We show that

$$-(-a) = a.$$

By definition, the additive inverse $-(-a)$ of $-a$, satisfies

$$-a + (-(-a)) = 0.$$

But by the definition of the additive inverse of a, we have

$$-a + a = 0.$$

Thus, by uniqueness of the additive inverse, it follows that

$$-(-a) = a.$$

Here are some examples.

(i) The additive inverse of 10 is the integer -10, since $10 + (-10) = 0$.
(ii) The additive inverse of 0 is 0. Since $0 + 0 = 0$, it follows that $-0 = 0$.
(iii) The additive inverse, $-(-7)$, of -7 is 7, since $-7 + 7 = 0$.

We will formally define the notion of positive and negative in Seminar 2. However, we want to point out now, that the examples above show that the additive inverse of an integer a can be positive or negative or zero. Resist the temptation to call $-a$ "negative a." As we have just seen, the additive inverse $-(-7)$ of the integer -7 is 7, which is not a negative number. The proper name for the additive inverse of the integer a is "*minus* a."

"Adding minus a to b" is precisely what we do when we subtract a from b. We define the operation of *subtraction* in $\mathbb{Z}$ as follows. If a and b are integers, then

$$b - a = b + (-a).$$

Thus subtraction is not a new operation. Subtracting a from b simply means adding the additive inverse of a to b. For example,

$$5 - 2 = 5 + (-2) = 3.$$

Another example is

$$-5 - 2 = -5 + (-2) = -7.$$

But, be alert, the operation of subtraction has few of the properties that addition has. For example, subtraction is neither associative:

$$(5 - 3) - 1 \neq 5 - (3 - 1)$$

nor commutative:

$$5 - 2 \neq 2 - 5.$$

Next, we investigate the rules for multiplication in the integers. The first three properties are analogous to those for addition.

(M1) Closure. If a and b are in $\mathbb{Z}$, then $a \cdot b$ is in $\mathbb{Z}$.

For example, the product of $22 \cdot 39 = 858$, an integer.

(M2) Associative Rule. If a, b and c are in $\mathbb{Z}$, then

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

For example, $(2 \cdot 2) \cdot 3 = 12$ and $2 \cdot (2 \cdot 3) = 12$. Thus, multiplication is associative, whereas, we saw that exponentiation and subtraction are not.

(M3) Commutative Rule. For integers a and b,

$$a \cdot b = b \cdot a.$$

So $5 \cdot 3 = 15$ and $3 \cdot 5 = 15$. Also, $(-67) \cdot 4 = -268$ and $4 \cdot (-67) = -268$.

The integers have a *multiplicative identity*. In other words, the equation $a \cdot x = a$, for all integers a, can be solved for x. Moreover, there is a unique such solution.

(M4) Multiplicative Identity. There is precisely one element in $\mathbb{Z}$, denoted 1, such that

$$a \cdot 1 = 1 \cdot a = a, \text{ for all } a \text{ in } \mathbb{Z}.$$

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Seminar Exercise. $\mathbb{Z}$ has a multiplicative identity 1. Consequently, it makes sense to ask if there are integers that have a multiplictive inverse. What do you think is the definition of a multiplicative inverse for an element a of $\mathbb{Z}$? Find the integers, if any, that have a multiplicative inverse.

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Comments on the Seminar Exercise. If a is an integer, a *multiplicative inverse for* a is the unique *integer* x that satisfies the equation

$$ax = 1.$$

The integer 1 has a multiplicative inverse 1, since $1 \cdot 1 = 1$. The integer -1 has a multiplicative inverse -1, since $(-1) \cdot (-1) = 1$. Note that although $2 \cdot (1/2) = 1$, the integer 2 does not have a multiplicative inverse because $1/2$ is not an integer. In fact, 1 and -1 are the only integers that have a multiplicative inverse in the set of integers.

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Finally, we have the important rule that links addition and multiplication. Moreover, this is the *only* rule that links addition and multiplication. Its importance is hard to overstate.

(D) Distributive Rule. If a, b and c are integers, then

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c);$$
$$(a + b) \cdot c = (a \cdot c) + (b \cdot c).$$

Observe that the second equation follows from the first by rule (M3). We look at two examples. The first is an example of left distributivity.

$$7 \cdot (-3 + 10) = (7 \cdot (-3)) + (7 \cdot 10) = (-21) + 70 = 49.$$

The next is an example of right distributivity.

$$(2 - 15) \cdot (-4) = (2 \cdot (-4)) + ((-15) \cdot (-4)) = (-8) + 60 = 52.$$

2. Roll Back, A Number Game of Chance

In this section we introduce a number game, called Roll Back, based on the rules of arithmetic. It has proved popular in the classroom.

Players: Roll Back can be played by groups of one to five students.

Supplies: One die, i.e., one of a pair of dice, for *each group* of players. A pencil and paper for *each player* in a group.

Object of the Game: The object of the game is to obtain a nonnegative balance by subtracting numbers from 100 based on rolls of the die.

Rules of the Game:

1. The players in each group take turns rolling the die which is rolled a total of five times. The players begin by writing the number 100 at the top of their papers.

2. After the first roll of the die, each player has a choice: either subtract from 100 the number rolled OR subtract from 100 the number rolled multiplied by 10. The result is called the first balance.

3. The most important rule of the game is that this balance and all subsequent balances must be nonnegative numbers.

4. After each of the four succeeding rolls of the die, each player has the same choice: either subtract from the previous balance the number rolled OR subtract from the previous balance the number rolled multiplied by 10. As each player makes these choices, the player must keep in mind that the balances must be nonnegative.

5. The student in each group with the nonnegative score closest to zero wins the game.

Here is an example.

Roll	Number Rolled	Choice	Balance
1	5	$5 \times 10 = 50$	100 - 50 =50
2	3	3	50 - 3 = 47
3	6	6	47 - 6 = 41
4	2	$2 \times 10 = 20$	41 - 20 = 21
5	3	3	21 - 3 = 18

After roll 3 and after roll 5, the player had no choice but to subtract the number rolled. After roll 2, the player could have chosen to subtract $3 \times 10 =$

30 from 50. Note that the players are doing some excellent mental, or pencil and paper, arithmetic as they make the choices required in the game.

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Seminar Exercise.

(i) Assume you have the same rolls of the die as in the sample game. Can you roll back to a score closer to zero than in the sample game?
(ii) Play another game of Roll Back.

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This game is one of many number games and classroom activities found in the authors' book *Trimathlon.* Games and classroom activities such as these play an important part of teaching arithmetic to elementary students.

3. Other Number Systems

The natural numbers,

$$\mathbb{N} = \{1, 2, 3, \dots, n, \dots\}$$

and the whole numbers,

$$\{0, 1, 2, 3, \dots, n, \dots\}$$

are subsets of the integers and inherit operations of addition and multiplication from $\mathbb{Z}$ because the sum and product of natural numbers are natural numbers, and the sum and product of whole numbers are whole numbers. Not all of the properties of these operations that hold for the integers also hold for the natural numbers and the whole numbers. We explore this fact in the next seminar exercise.

Note. If a set does not satisfy closure for a certain operation, then we do not discuss other properties for that operation on the set. If a set does not have an identity for a certain operation, then we do not discuss inverses for that operation.

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Seminar/Classroom Activity. For the following number systems, tell which of the rules of arithmetic (A1)–(A5), (M1)–(M4), and (D) hold and which do *not* hold. In the case of subsets of $\mathbb{Z}$, assume that addition and multiplication on the subset are the same as those in $\mathbb{Z}$. Justify your answers.

(i) The natural numbers $\mathbb{N}$.
(ii) The whole numbers.
(iii) The set $\mathcal{E}$ of even numbers.
(iv) The set $\mathcal{O}$ of odd numbers.
(v) The set $\mathcal{S}$ of squares of whole numbers.
(vi) This set is not a subset of $\mathbb{Z}$. Let $F = \{\mathcal{E}, \mathcal{O}\}$. Define two operations $+$ and $\cdot$ on F by means of the following operation tables.

$+$	$\mathcal{E}$	$\mathcal{O}$
$\mathcal{E}$	$\mathcal{E}$	$\mathcal{O}$
$\mathcal{O}$	$\mathcal{O}$	$\mathcal{E}$

$\cdot$	$\mathcal{E}$	$\mathcal{O}$
$\mathcal{E}$	$\mathcal{E}$	$\mathcal{E}$
$\mathcal{O}$	$\mathcal{E}$	$\mathcal{O}$

Read the table in the order *row* by *column*. For example, the sum of $\mathcal{E}$ and $\mathcal{O}$ is found at the intersection of row 2 and column 3, and the product of $\mathcal{O}$ and $\mathcal{E}$ is found at the intersection of row 3 and column 2.

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Comments on the Seminar/Classroom Activity.

(i) $\mathbb{N}$ does satisfy rules (A1)–(A3) and (M1)–(M4) and (D). $\mathbb{N}$ does not satisfy rule (A4). For there is no natural number x that satisfies the equation $1 + x = 1$. (We do not discuss additive inverses, since $\mathbb{N}$ does not have an additive identity.)

(ii) The whole numbers do satisfy (A1)–(A4) and (M1)–(M4) and (D). They do not satisfy rule (A5). There is no whole number x that satisfies the equation $1 + x = 0$.

(iii) The set $\mathcal{E}$ does satisfy (A1)–(A5) and (M1)–(M3) and (D). $\mathcal{E}$ does not satisfy rule (M4). For there is no even number x that satisfies the equation $2 \cdot x = 2$.

(iv) The set $\mathcal{O}$ does not satisfy rule (A1) because the sum of two odd numbers is even, so we do not discuss rules (A2)–(A5) or (D). The set $\mathcal{O}$ does satisfy rules (M1)–(M4.)

(v) The set $\mathcal{S}$ does not satisfy rule (A1) because the sum of two squares need not be a square. For example, $1^2 + 1^2 = 2$ is not a square. All multiplicative properties are satisfied.

(vi) The set F satisfies all of the rules (A1)–(A5), rules (M1)–(M4) and (D). The element $\mathcal{E}$ of F is the additive identity and the element $\mathcal{O}$ is the multiplicative identity.

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4. Arithmetic in the Integers, Part II

In Section 1, we described the rules for addition and multiplication in the integers. In this section, we will deduce some familiar arithmetical facts from the rules (A1)–(A5), (M1)–(M4) and (D). We begin with a formal definition of a binary operation on a set S. A *binary operation on* S or *operation on* S, for short, is a rule $\star$ that associates to every pair (hence the word "binary") of elements (a, b) in any set S another element, denoted $a \star b$, *in* S. Note that the definition of binary operation includes closure. As we have seen, addition and multiplication of pairs of integers are binary operations on $\mathbb{Z}$. Subtraction is a binary operation on $\mathbb{Z}$ but not on $\mathbb{N}$ or on the set of whole numbers, because neither $\mathbb{N}$ nor the set of whole numbers is closed under subtraction. The number system F introduced in the Seminar/Classroom Activity in Section 3 is very interesting because it has only two numbers $\mathcal{E}$ and $\mathcal{O}$. The addition and multiplication tables given in that activity define two binary operations on the number system F.

Note. From now on, we will use any of the notations ab, $(a)(b)$ or $a \cdot b$ to denote the product of the integers a and b. The choice is usually dictated by a wish for clarity.

When we solve certain equations involving addition, one of the procedures we might like to apply is cancellation. Should this be another rule, or does it follow from the ones we have?

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Seminar Exercise. Show how the property (AC), defined below, follows from rules (A1)–(A5).
(AC) Additive Cancellation. If

$$c + a = c + b, \quad \text{for} \quad \text{integers } a,\ b, \ \text{ and } c,$$

then

$$a = b.$$

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Comments on the Seminar Exercise. We verify Property (AC) by adding the additive inverse, $-c$, of c to both sides of the given equation:

$$-c + (c + a) = -c + (c + b).$$

By additive associativity (A2), we have:

$$(-c + c) + a = (-c + c) + b.$$

By (A5), $-c + c = 0$, and the result is

$$0 + a = 0 + b.$$

By (A4), $0 + a = a$, and $0 + b = b$, so the equality $0 + a = 0 + b$ implies that

$$a = b.$$

Thus, the additive cancellation property, (AC), follows from the rules (A1)–(A5).

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Next, we look at multiplication by the additive identity 0.
Multiplication by 0. For all integers a,

$$a \cdot 0 = 0 \cdot a = 0.$$

Since 0 is the additive identity and the property we are to prove is one involving multiplication, we expect that the distributive property (D) which links addition and multiplication will play a role. First we apply (A4) in an unusual way: we "replace 0 by $0 + 0$". By (A4),

$$0 + 0 = 0.$$

Thus,

$$a \cdot (0 + 0) = a \cdot 0.$$

We apply (D) to the left side of the equation above to obtain

$$a \cdot (0 + 0) = (a \cdot 0) + (a \cdot 0).$$

It follows from the equation $a \cdot (0 + 0) = a \cdot 0$ that

$$(a \cdot 0) + (a \cdot 0) = a \cdot 0.$$

By (A4), we may add 0 to the right hand side above and retain the equality:

$$(a \cdot 0) + (a \cdot 0) = (a \cdot 0) + 0$$

Finally, we apply (AC), and we have the desired conclusion:

$$a \cdot 0 = 0.$$

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Seminar Question. Are there any pairs of integers (a, b) for which $a + b = a \cdot b$?
Challenge. If so, find all such pairs.

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Comments on the Seminar Question. Since $0+0 = 0 \cdot 0$, and $2+2 = 2 \cdot 2$, the answer to the question is "yes." The argument showing that these are the only such pairs is left for you to think about.

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Properties that involve an integer of the form "minus a," that is, $-a$, must be interpreted properly. Recall that $-a$ is defined to be the unique additive inverse of a. In other words, $-a$ is the integer with the property that

$$a + (-a) = 0 = (-a) + a,$$

and it is the only such integer. Consequently, to show that a certain integer b presented to us is equal to $-a$, for some integer a, what we must do is show that when we add a to b, the sum is zero. Here are three such examples.
(PAI) Properties of Additive Inverses in $\mathbb{Z}$.

(i) For all integers a,

$$(-1) \cdot a = -a.$$

(ii) For all integers a and b,

$$a(-b) = -(ab) \text{ and } (-a)b = -(ab).$$

(iii) For all integers a and b,

$$(-a)(-b) = ab.$$

In words, property (i) states that the integer $(-1) \cdot a$, the product of the additive inverse of the multiplicative identity and a, is the additive inverse of a. Consequently, we must show that when we add $(-1) \cdot a$ to a, we obtain the additive identity 0. Since $(-1) \cdot a$ is a product and we are asked to use it additively, we expect distributivity to come into play, and it does. First, we apply the rule (M4) for the multiplicative identity 1,

$$a + (-1) \cdot a = 1 \cdot a + (-1) \cdot a.$$

Next, we apply (D) to write

$$1 \cdot a + (-1) \cdot a = [1 + (-1)] \cdot a.$$

By (A5),

$$[1 + (-1)] \cdot a = 0 \cdot a.$$

We apply the multiplication by 0 property, to obtain

$$0 \cdot a = 0.$$

Thus, we may conclude that

$$a + (-1) \cdot a = 0,$$

and

$$(-1)a = -a.$$

In words, property (ii) states that the integer $a(-b)$ is the additive inverse of ab, and that the integer $(-a)b$ is the additive inverse of ab. To show that $a(-b)$ is the additive inverse of ab, we must show that $a(-b) + ab = 0$. We first apply distributivity, (D),

$$a(-b) + ab = a(-b + b).$$

By (A4),

$$-b + b = 0,$$

and, by the multiplication by 0 property,

$$a \cdot 0 = 0.$$

Thus, we may conclude that

$$a(-b) + ab = 0 \text{ and } a(-b) = -(ab)$$

A similar argument, which we leave to you, shows that $(-a)b = -(ab)$.

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Seminar Exercise. Show that

$$(-a)b = -(ab).$$

Then verify (iii): $(-a)(-b) = ab$.

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Comments on the Seminar Exercise. We must show that $(-a)b+(ab) = 0$. By (D),

$$(-a)b + ab = (-a + a)b.$$

By (A4),

$$-a + a = 0.$$

By the multiplication by 0 property,

$$0 \cdot b = 0.$$

Thus,

$$(-a)b + ab = 0 \text{ and } (-a)b = -(ab)$$

To prove (iii), we must show that $(-a)(-b) + -(ab) = 0$. But we have just shown that $-(ab) = (-a)b$, so, by the same reasoning as above,

$$(-a)(-b) + -(ab) = (-a)(-b) + (-a)b = (-a)[-b + b] = (-a)(0) = 0.$$

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Reference

Sally, J. and Sally, Jr., P., *Trimathlon*, A. K. Peters, Ltd., 2003.