Preliminaries

What is a Mathematical Circle?

When people talk about mathematics in everyday life, they often mean arithmetic skills. Likewise, when parents talk about their children learning mathematics, they often judge their success in terms of proficiency in solving standardized test problems. Mathematical circles have very little to do with these things.

Mathematical circles do not teach arithmetic skills and do not train for tests. Instead, they use mathematics as a powerful tool for encouraging logical reasoning, developing creativity, and enhancing students' ability to analyze and solve complex problems. The benefits of studying in a math circle are much broader than the expansion of mathematical knowledge. Participation in a circle helps students discover and develop their talents and inspires in them a lifelong love for mathematics and related sciences.

The subjects covered in mathematical circles normally go beyond the school curriculum, and are exciting and fun. The problems and discussion topics are selected to stimulate students' creative abilities and develop their problem-exploration skills and mathematical reasoning. A lot of class time is usually spent on solving and discussing carefully selected series of problems. These discussions are accompanied by presentations of related theoretical material. Quite often, theoretical facts come to students naturally in the context of problems they solved. Mathematical games and contests are an important part of the curriculum as well.

A key role of mathematical circles is in providing a place where mathematically motivated students can meet similar-minded peers. Participation in a circle helps students become members of a community and make friends who share their interest in learning.

Circles are for children who have curiosity and interest in math and science, enjoy intellectual challenges, and are ready to explore mathematics beyond school curriculum boundaries. Most mathematical circle students will probably not become professional mathematicians, choosing very different career paths instead. However, mathematics will remain a good friend to them throughout their formal education and beyond. Mathematical culture, problem-solving abilities, and the analytical skills fostered in math circles are powerful tools indispensable for success in many professions.

About This Book

In recent years, the popularity of circle-style extracurricular mathematics in United States has been steadily growing. Some parents see this type of math as a way to develop their children's creative thinking skills. Others treat circle math as a means for developing appreciation of the subject and stimulating interest in learning. Many parents want to boost their children's mathematical knowledge and get them involved in mathematical competitions and Olympiads. In addition, more and more universities are considering the idea of running mathematical circles as their contribution to educating schoolchildren.

As a result, there is a lot of interest among parents and math professionals in organizing a math circle or in learning how to teach circle-style mathematics. The process of self-education in this area can be quite time-consuming; a potential teacher usually needs to read through several books in order to form his or her teaching principles and lay out the curriculum for the circle.

This book tries to speed up the process. It has everything that is needed to run a circle for a full year, distributed among 29 lessons: materials to be presented in class, sets of problems to work on, and contests and games to play with the students. In addition, the book tries to impart some of the know-how needed in running a mathematical circle by providing advice on organizing sessions, presenting the material, and avoiding typical mistakes.

You can use the book in several ways:

– As the backbone of the circle curriculum.

– As a source of lectures and problems on the topics that you find suitable for your group.

– As a collection of problems, contests and games.

For the past decade, the author has been actively involved in teaching a number of mathematical circles in the United States. This book is based on her experience and on the compilation of materials from these circles.

The book is intended for teaching students in grades 5 to 7. Children at this age make great math circle students. It is an age of curiosity and of openness to learning: the discoveries students make in class and the triumphs of having a problem solved make them proud of their achievements. The learning skills and the thinking habits acquired by students at this time stay with them for life and will come in handy for future learning. The principles of problem solving developed in math circles will make it possible to find solutions to extremely challenging problems later on.

Teaching extracurricular math to students of this age is a rewarding experience for the teacher as well. It is fun to show young students the joys of math and to observe how they grow and develop their abilities.

About the Author

My name is Anna Burago. I grew up in St. Petersburg, Russia, one of the centers of Russian mathematical culture and a nurturing place to grow up in for a math-minded child.

I started to attend a math circle when I was in the fifth grade. Since then, extracurricular mathematics has been the passion of my life. After I graduated from a specialized high school with advanced math classes, I went on to major in mathematics at the St. Petersburg State University.

Participating in a math circle was one of the happiest experiences of my school years, not only because of what I learned but also because of the new friends I met. Great teachers from my math school and math circles had always inspired awe and excitement in me. Consequently, during all the years of my university life, I was an enthusiastic teacher of a math circle of my own.

In the United States, I've spent many years working as a software developer (I hold M.S. degrees in both mathematics and computer science). The idea to start a mathematical circle came to me in 2002, when my oldest son Timothy became a fifth grader. My husband Andrei is a math circle graduate and enthusiast as well. We were both sure that, given our son's interests and personality, a math circle experience would be very beneficial to him. However, giving him individual lessons did not work well. When we started looking for brain-challenging math classes in the area where we lived (Seattle), we found nothing but centers preparing students for tests.

Fortunately, we had several friends in the area who wanted to provide their children with the benefits of math circle education. Therefore, Andrei, our friend Alexander Gil, and I took a deep breath, put up a whiteboard in our family room, and started a mathematical circle. We invited children of our friends, friends of our children, children of friends of our friends and friends of friends of our children. To our surprise and pride, this first circle lasted for 8 years, coming to an end only when most of its students graduated from high school and entered college. Through those years, the size of our circle fluctuated somewhat, but a core group of students stayed with us more or less throughout. Now several of our graduates are involved in teaching math circles as well.

Three years after the first circle was founded, we joined forces with some other parents and started another one. Within the next few years two more circles sprang up, all powered by the desire of the parents to give their growing children the circle experience and a circle-triggered brain boost. Currently, I am teaching another brand-new circle for fifth and sixth

Session 13: Meet the Cube. First Lesson in 3D Geometry

This session and the next deal with three-dimensional (3D) geometry. You may be surprised that we introduce this topic without discussing two-dimensional (2D) geometry first. But we live in a three-dimensional world, and the ability to see and operate in this world comes to us naturally. It is also very important for us to know how to *analyze* this world and how to *represent* it on two-dimensional surfaces (on paper and on computer screens).

Therefore, it is good for children to practice three-dimensional thinking: the ability to visualize and manipulate objects in space and the ability to relate 3D objects to their 2D images.

Three-dimensional geometry is a treasure trove of topics that are interesting and challenging for children and, at the same time, require no special knowledge. Working on these topics develops spatial intuition, 3D reasoning skills, and abstract spatial modeling abilities.

Lastly, 3D geometry is largely overlooked by schools and parents. You would be surprised to learn how little practice in spatial visualization most students receive. Even the best-trained equation-solvers have difficulties imagining a hexagonal cross section of a cube.

Thus, in the lessons to follow, we will start exploring the rich world of three-dimensional geometry. We will familiarize ourselves with several 3D objects, learn how to work with cross sections, and, overall, try to get into a 3D-thinking mode.

The topic of today's investigation is a 3D cube. A cube is one of the easiest 3D shapes to think about and to visualize; we are surrounded by various incarnations of it in our everyday life.

Teaching supplies for this session:

– A large cube for the teacher.

– A cube for each student. (These cubes could be anything: children's building blocks, dice, etc. I use my daughter's ABC cubes.)

– A couple of large-size paper cutouts of cube nets (see page 100).

– Several pairs of scissors (for students who prefer to have paper cutouts while working on cube net problems).

– Enough copies of the class handout. (In addition to the regular problem set, this lesson contains a handout for in-class independent work, to give students extra practice in working with 3D cubes and cube nets.)

For Teachers: This is the next session after the math Olympiad, so some matters were left hanging — on the day of the Olympiad there was no time to discuss the problem set from two classes ago, and we have not talked about the Olympiad problems either. The question is whether to present the solutions to these problems or to ignore them.

My experience is that students are not too eager to look back. If you start discussing all these past problems, you'll get a very boring lesson. A reasonable course of action is to discuss the juiciest Olympiad problems and ignore the rest. As for the problem set, the problems that are deemed important can be reformulated and reassigned.

Math Warm-up

Warm-up 1. A pilgrim traveling from Avignon to the Vatican met seven peasants heading back from the Vatican to Avignon. Each peasant had a basket with a goose and a sack with three cats. How many creatures in total were on their way to the Vatican?

Warm-up 2. Why would a London barber prefer to give a haircut to two Frenchmen rather than to one German?

Discussion of the Day: Cubes

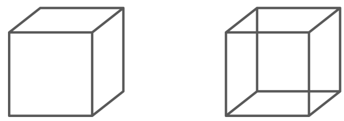
I initiate the discussion by asking the students to define a cube. How would they describe a cube to an alien from another planet? (This is not a rigorous exercise. In my class, we had assumed that aliens are already familiar with 2D geometry concepts. Therefore, we ended up describing a cube as a shape with six identical square faces.)

This is the right moment to hand out the cubes to the students and start discussing basic facts about cubes. I ask the children to name and count the main parts of a cube. How many faces does it have? How many vertices? How many edges?

Next, let's spend some time on visual exploration. Look at the cube. How many faces can you see simultaneously? Are you able to position your cube in such a way as to see three faces? Yes, easily. How about two faces? One face? Yes, this is all possible. Can we see four or more faces at the same time? No, this cannot be done.

The question that comes up next is how to draw a cube on a piece of paper — a flat 2D surface. Do we sketch the cube in such a way that just one of its faces is visible? This way, the drawing of the cube would look like a square to us. How about two visible faces? This picture would look like two quadrilaterals joined along one side. What if we draw it with all three faces visible? This view would likely be the most useful one...

Hence we usually draw a cube in such a way that three of its faces are visible. This is conventionally done as in the first drawing here:

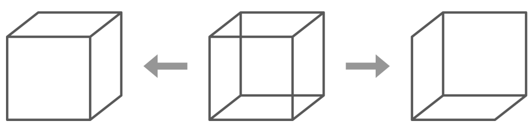


The second drawing shows what's called a *wireframe model* of the cube: what's left of the cube if you remove the faces and the interior, and leave only the edges. (Alternatively, you can think of the faces as transparent.)

What are the differences between these two images? Which one would work better for us?

The solid cube on the left is very realistic, and it is perfect for a drawing lesson. However, it has one disadvantage: it is impossible to see the hidden edges and back faces of this cube. Therefore, in our math classes we generally stick to the wireframe model.

But the wireframe drawing in the figure above is not perfect either. Can you tell which is the front face? Yes and no, because there are two possible answers:

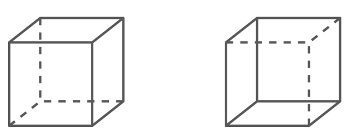


Depending on where you focus your eyes, you can see two different cubes: one with its front face on the lower left, and one with its front face on the upper right.

For Teachers: To help the students visualize these two options, make two wireframe drawings on the board and shade the front faces of the two cubes in different colors.

A lot of optical illusions take advantage of such ambiguities; they involve pictures that allow for two different interpretations. Unfortunately for us, it is not mathematically correct to be able to interpret the same picture in two different ways. Therefore, we need to somehow correct our wireframe drawing — we need to indicate which of the edges are the hidden ones.

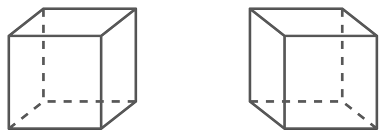
The best solution is to draw the invisible edges with dashed lines instead of solid lines. So instead of the ambiguous wireframe drawing in the middle of the preceding illustration, we use one of the following drawings:



Here the solid lines indicate the edges that are visible in a solid view, and the dashed lines indicate the hidden edges. This is how we are going to draw our cubes and other 3D objects.

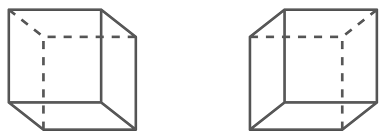
Exercise 1. For this exercise, every student should have a cube handy.

(a) Grab a cube; try to position your hand and orient the cube in such a way as to be able to view the cube as it is displayed on the left picture below:



(b) Repeat the same exercise for the right picture.

Exercise 2. Repeat the same exercise for the drawings shown here:



For Teachers: These apparently innocent exercises could take a big chunk of your class. You might need to spend a lot of time one-on-one with younger students: explaining how to interpret the pictures and demonstrating how to hold the cube. However, the result is worth the time; your students need this kind of spatial visualization practice.

Cube Nets

Our next discussion topic is about two-dimensional figures that can be folded into three-dimensional objects. For this conversation, you will need two paper cutouts, in the shapes shown below. Each cutout consists of six identical squares and can be made of a single sheet of A4 paper.

Show the first cutout to the students. Can this shape be folded into a cube in such a way that each square becomes a face of the cube? The answer — hopefully generated by the students — is positive. Here's how you do it. (It helps to score the folds ahead of time.) First, fold the long strip of four squares into a vertical four-sided "tube", with one flap hanging from the bottom of the tube and the second flap attached to the top. Next, close the flaps like lids: these will form the top and bottom faces of the cube.

We just folded a two-dimensional shape into a three-dimensional cube. The question of what two-dimensional shapes can be folded into 3D objects generates a lot of interesting problems. Before we start exploring some of them, we introduce some definitions.

Definition 1. A *net* is a two-dimensional shape that can be folded into a three-dimensional body.

Definition 2. A *cube net* is a two-dimensional shape that can be folded into a cube.

For Teachers: For now, we will assume that all our cube nets are composed of six identical squares; that is, they (conceptually) come from cutting a cube open along several of its edges and then unfolding it.

While it is possible to create more complex cube nets (if cuts are made across the faces of a cube), we will postpone working with those.

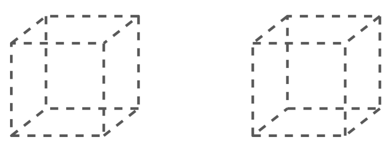
Could any figure made of six identical squares connected side-to-side be folded into a cube? The answer is no; for example, the second cutout shown on the previous page, composed of six squares arranged into a long strip, cannot be folded into a cube. Why? We can assume that the strip starts out with the long edges horizontal and the plane of the paper vertical. Then no matter how you fold along the lines, all the faces will remain vertical, because the hinges are vertical. So none of the faces can end up as the bottom or the top of the cube.

For Teachers: We chose this starting orientation to make the proof easier to state, but in fact the starting orientation does not matter: any recipe for folding the cutout into a cube can be reinterpreted as a recipe starting in the chosen orientation.

Now we turn to independent work. It is important for students to have as much 3D practice as possible. This lesson comes with an in-class handout and a home problem set. The handout contains exercises on both the 2D representation of cubes and on the cube nets.

"Meet the Cube" Exercises

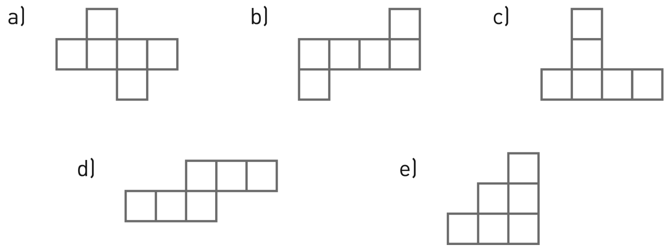
Exercise 1. Complete each of the two identical cube templates below into a different view of a cube:



For the first sketch, position a model of a cube so that you can see the front, right and top faces of it. (You can place this cube on your desk to the left of you). Draw the cube on the template the way you see it.

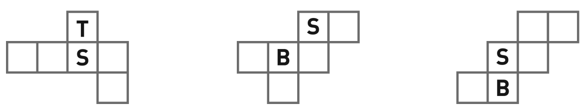
For the second sketch, position the cube so that you can see the front, left and bottom faces of it. (You can hold the cube in your hand in front of you, slightly to your right and above the eye level). Draw the cube on the template the way you see it.

Exercise 2. Which of the following 2D shapes are cube nets (that is, can be folded into cubes)?



Exercise 3. Marian made several 3D paper models of the cube from cutouts. On each assembled cube, she labeled the faces with letters: T for the top, B for the bottom, and S for all the side faces. Next, Marian unfolded the cutouts to store them.

Later Marian's younger sister, Bella, erased some of the letters. Here is what they looked like at that point:



Please restore the missing letters.

Session 13 Problem Set

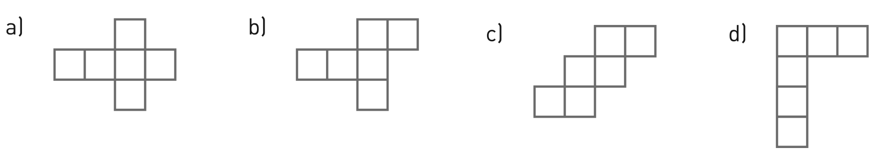
Problem 1. **(a)** Out of 10 coins, one is counterfeit: it is lighter than the real ones. Show how to find the counterfeit coin in three weighings on the balance scale with two pans.

(b) Out of 16 coins, one is counterfeit; it is lighter than the real ones. Show how to find the counterfeit coin in three weighings.

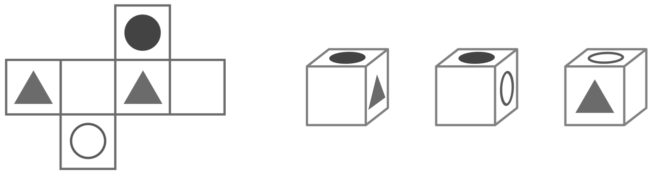
(c) Out of 27 coins, one is counterfeit: it is lighter than the real ones. Show how to find the counterfeit coin in three weighings.

Problem 2. Mister Buckley, the baker, has a big sack of buckwheat flour. He also has a balance scale with two pans. How can he measure out 7 pounds of flour in three weighings using one 1-pound weight?

Problem 3. Which of these shapes are cube nets (that is, can be folded into cubes)?



Problem 4. Which of the cubes below could have been folded from the cube net sketched on the left?



Problem 5. **(a)** Is it possible to cut a square into 10 smaller squares (not necessarily of equal size)? If so, show how. If not, explain why. (Squares that are subdivided into smaller ones don't count.)

(b) Same question for 11 squares.

Problem 6. Snow White decided to make a quilt. She took a big square piece of fabric and cut it into 4 equal squares. She left them on the table and went to the kitchen. The first dwarf, who happened to pass by the table, picked one square at random. He cut it into four equal squares and placed them back on the table. All the other dwarfs (the second, the third, the fourth, the fifth, the sixth and the seventh) did the same thing: each picked a random piece of fabric and cut it into four smaller squares. How many pieces of fabric were there on the table when Snow White came back to the room?

Problem 7. A lame but merry cricket jumps along a straight line. On every jump, it leaps 3 inches to the right or 5 inches to the left.

(a) Is there a way for the cricket to end up 1 inch to the right of its starting point? How about 1 inch to the left? If the answer is yes, show the sequence of jumps. If it is no, explain why.

(b) The cricket made 20 jumps. Is it possible for it to end up 1 inch to the right of the starting point?

(c) The cricket made 23 jumps. Is it possible for it to end up 10 inches to the left of the starting point?