

# Contents

Acknowledgments	xi
Preliminaries	1
What is a Mathematical Circle?	1
About This Book	2
About the Author	3
Why I Wrote This Book	4
Target Age Groups	5
<b>Part 1. Session Plans</b>	<b>7</b>
Introduction	9
A Few Words about Problem Sets	10
Session 1: How to Solve a Problem	11
Before the Session	11
The Session Plan	12
Introductions and Math Warm-up	12
What is a Mathematical Circle?	13
Discussion of the Day: “What Does it Mean to Solve a Problem?”	13
A Few Words about Problem Sets and Homework	19
Session 1 Problem Set	19
Session 2: Knights and Liars	23
Math Warm-up	23
Problem Set From the Previous Class	24
Discussion of the Day, Part 1: Cause and Effect Problems	24
Discussion of the Day, Part 2: Knights-and-Liars Problems	25
Session 2 Problem Set	28
Session 3: How to Turn Lies into Truth	31
Math Warm-up	32
Discussion of the Day: Negation of Logical Statements	33
Session 3 Problem Set	34

Session 4: Mathematical Auction	37
Math Warm-up	37
The Game of Mathematical Auction	37
Mathematical Auction Problems	40
Session 4 Problem Set	41
Session 5: Word Problems and Common Sense	43
Math Warm-up	43
Discussion of the Day: “Draw a Picture—Solve the Problem!”	43
Discussion of the Day, Part 2: Working Together	46
Session 5 Problem Set	47
Session 6: More Word Problems	51
Math Warm-up	51
Discussion of the Day: Of Pigs and Hens	52
Session 6 Problem Set	53
Session 7: Odd and Even Numbers I. Magic Paper Cups	57
Math Warm-up	57
Discussion of the Day: Odd and Even Numbers. Parity	58
Session 7 Problem Set	60
Session 8: Odd and Even Numbers II. Definitions and Properties	63
Math Warm-up	63
Discussion of the Day: How to Define Odd and Even Numbers	63
Parity of Sums and Differences	65
Parity of a Product	69
Session 8 Problem Set	69
Session 9: Halloween Math Hockey I	71
Math Warm-up	71
Math Hockey	71
Math Hockey I Problems	72
Session 9 Problem Set	74
Session 10: Odd and Even Numbers III. Alternations	77
Math Warm-up	77
Discussion of the Day: Alternations	77
Session 10 Problem Set	81
Session 11: Weighings and Counterfeit Coins	85
Math Warm-up	85
Discussion of the Day: Balance Scales and Counterfeit Coins	86
Session 11 Problem Set	89
Session 12: Mathematical Olympiad I	91
Math Warm-up	91
Event of the Day: Math Olympiad	92

Contents	vii
Math Olympiad I. First Page of Problems	93
Math Olympiad I. Second Page of Problems	93
Math Olympiad I. Additional Problems	94
Session 13: Meet the Cube. First Lesson in 3D Geometry	97
Math Warm-up	98
Discussion of the Day: Cubes	98
Cube Nets	100
“Meet the Cube” Exercises	101
Session 13 Problem Set	102
Session 14: Cross Sections. Second Lesson in 3D Geometry	105
Math Warm-up	105
Discussion of the Day: Cross Sections	106
Session 14 Problem Set	110
Session 15: Mathematical Auction	113
Math Warm-up	113
Event of the Day: The Game of Mathematical Auction	113
Mathematical Auction Problems	114
Session 15 Problem Set	115
Session 16: Combinatorics I	117
Math Warm-up	117
Discussion of the Day: Introduction to Combinatorics	117
Session 16 Problem Set	124
Session 17: Combinatorics II	127
Math Warm-up	127
Discussion of the Day: The Slot Rule and Permutations without Repetitions	127
Session 17 Problem Set	134
Session 18: Mathematical Hockey II	137
Math Warm-up	137
Event of the Day: Math Hockey Game	137
Math Hockey II Problems	138
Session 18 Problem Set	139
Session 19: Numerical Puzzles I. Runaway Digits	141
Math Warm-up	141
Discussion of the Day: Runaway Digits	142
Session 19 Problem Set	144
Session 20: Numerical Puzzles II. Encrypted Problems	147
Math Warm-up	147
Discussion of the Day: Encrypted Problems	147
Session 20 Problem Set	150

Session 21: Mathematical Olympiad II	153
Event of the Day: Math Olympiad	153
Math Olympiad II. First Page of Problems	154
Math Olympiad II. Second Page of Problems	155
Math Olympiad II. Additional Problems	156
Session 22: Divisibility I. Definition and Properties	157
Math Warm-up	157
Discussion of the Day: Divisibility	158
How Many Factors?	159
Divisibility: More Definitions	159
Notation for Divisibility	160
Divisibility Properties	161
Session 22 Problem Set	163
Session 23: Divisibility II. Prime Numbers and Prime Factorization	165
Math Warm-up	165
Discussion of the Day: Prime Numbers and Divisibility	165
Prime Factor Decomposition	166
Prime Factorization Tree	168
Prime Factorization and Divisibility	168
The Prime Factorization of a Product	170
Session 23 Problem Set	171
Session 24: Mathematical Auction	173
Math Warm-up	173
The Game of Mathematical Auction	173
Mathematical Auction Problems	174
Session 24 Problem Set	174
Session 25: Divisibility III. Divisibility Rules	177
Math Warm-up	177
Discussion of the Day: Divisibility Rules	177
Divisibility and Cryptography	181
Session 25 Problem Set	182
Session 26: Divisibility IV. Relatively Prime Numbers	185
Math Warm-up	185
Discussion of the Day: Relatively Prime Numbers	185
Session 26 Problem Set	188
Session 27: Mathematical Games of Strategy I	191
Math Warm-up	191
Discussion of the Day: Games and Winning Strategies	191
Session 27 Problem Set	195
Session 28: Mathematical Games of Strategy II	197

Contents	ix
Math Warm-up	197
Discussion of the Day: Complements	197
Pseudogames	199
Session 28 Problem Set	200
Session 29: Mathematical Olympiad III	203
Math Warm-up	203
Event of the Day: Math Olympiad	204
Math Olympiad III. First Page of Problems	204
Math Olympiad II. Second Page of Problems	205
Math Olympiad II. Additional Problems	206
<b>Part 2. Mathematical Contests and Competitions</b>	<b>209</b>
Mathematical Contests	211
Mathematical Auction	213
Rules of Mathematical Auction	213
A Sample Round of Mathematical Auction Game	215
Team Work	216
Advice for the Teacher	216
Example Math Auction Problems	217
Mathematical Hockey	219
Mathematical Hockey Rules	219
Advice for the Teacher	220
Mathematical Olympiads	223
Types of Olympiads	223
Olympiad Problems	224
Planning for an Oral Olympiad	225
Olympiad Rules	225
Olympiads in This Book	226
Awards and Prizes	226
Short Entertaining Math Games	229
Jotto and Math Jotto	229
Nim	229
Black Box	230
<b>Part 3. More Teaching Advice</b>	<b>233</b>
How to be a Great Math Circle Teacher	235
Math Circle Day-to-Day	239
A Typical Session in Our Circle	239
A Math Entertainment Session in Our Circle	241
How to Organize a Session in Your Circle	242

A Few More Ideas on the Layout of a Session	243
How to Keep a Circle Journal	243
How to Organize Problem-Solving Time	244
More Questions?	247
<b>Part 4. Solutions</b>	249
Session 1: How to Solve a Problem	251
Session 2: How to Solve a Problem and Explain a Solution	254
Session 3: How to Turn Lies into Truth	257
Session 4: Mathematical Auction	259
Session 5: Word Problems and Common Sense	260
Session 6: More Word Problems	262
Session 7: Odds and Evens – Magic Paper Cups	266
Session 8: Odds and Evens – Definitions and Properties	269
Session 9: Math Hockey I	271
Session 9: Problem Set Solutions	272
Session 10: Odds and Evens – Alternation	274
Session 11: Weighing and Fake Coins	276
Session 12: Math Olympiad I	279
Session 13: Meet The Cube	283
Session 14: Cross Sections	285
Session 15: Problem Set Solutions	288
Session 16: Combinatorics I	291
Session 17: Combinatorics II	295
Session 18: Math Hockey II	298
Session 18: Problem Set Solutions	299
Session 19: Runaway Digits	301
Session 20: Encrypted Problems	303
Session 21: Math Olympiad II	305
Session 22: Divisibility I	310
Session 23: Divisibility II	313
Session 24: Problem Set Solutions	316
Session 25: Divisibility III	318
Session 26: Divisibility IV	321
Session 27: Mathematical Games of Strategy I	323
Session 28: Mathematical Games of Strategy II	326
Session 29: Math Olympiad	328
Bibliography	335