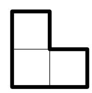
Problems

Year 2011

6th Class

1. "You're not ready to see this," said Baba-Yaga to her 33 students. "Close your eyes!," she commanded. All the boys and one third of the girls closed their right eyes; all the girls and one third of the boys closed their left. How many students were able to get a peek, despite Baba-Yaga's demand?

2. Partition a 6×6 chessboard with triominos (see the figure) so that no two adjacent triominos forms a 2×3 rectangle.



3. Before a soccer game between North and South, there were five predictions:

a) There won't be a draw.
b) North will score against South.
c) North will win.
d) North won't lose.
e) Three goals were made, in total.

After the match, it turned out that exactly three of these predictions turned out to be true. What was the final score?

4. Find all solutions of the following cryptogram, where each distinct letter encodes a distinct digit.

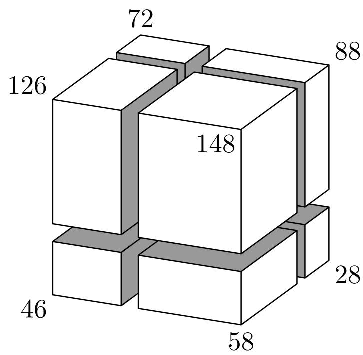
$$\mathrm{I} + \mathrm{HE} + \mathrm{HE} + \mathrm{HE} + \mathrm{HE} + \mathrm{HE} + \mathrm{HE} + \mathrm{HE} + \mathrm{HE} = \mathrm{US}.$$

5. Dragon imprisoned six gnomes in his cave and said, "I have seven caps, colored with each color of the rainbow. Tomorrow morning I will blindfold you and put a cap on each of you, and hide the last cap. Then I will take the blindfolds off, and you'll be able to see what color everyone else is wearing, but you cannot talk to one another. Then, without letting the others hear, tell me the color of the hidden cap. If

at least three of you guess correctly, I'll free you all. If not, I'll eat you for lunch."

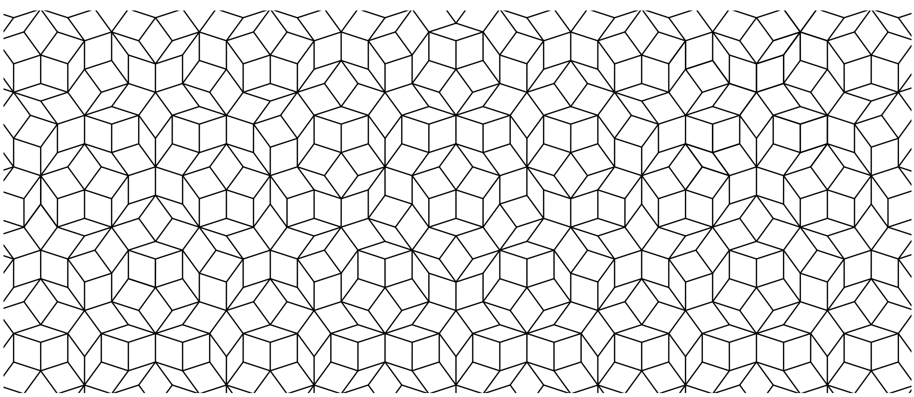
How can the gnomes come up with a strategy that saves them?

6. A wooden block is divided into eight smaller blocks by three cuts. In the figure, the areas of the seven visible blocks are labeled. What is the area of the eighth block?

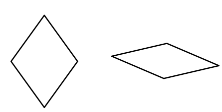


7th Class

1. Here is a fragment of a mosaic:



The mosaic is formed from two types of rhombuses: fat and skinny, as shown on the right. Mark out a section of this mosaic, consisting of contiguous pieces, of which exactly three are fat and eight are skinny rhombuses. (The region must be contiguous; it cannot be made of two separate pieces.)



2. Along the path between Homer and Marge's cottages grew a line of flowers containing 15 peonies and 15 tulips. Heading from her cottage to visit Homer, Marge watered all the flowers one-by-one as she

encountered them. After the 10th tulip, the water ran out, and 10 flowers remained dry.

The next day, heading from his cottage to visit Marge, Homer picked all the flowers one-by-one as he encountered them. Picking the 6th tulip, he decided that the had enough for a bouquet. How many flowers were left growing along this path?

3. Before a soccer game between North and South, there were five predictions:

a) There won't be a draw.
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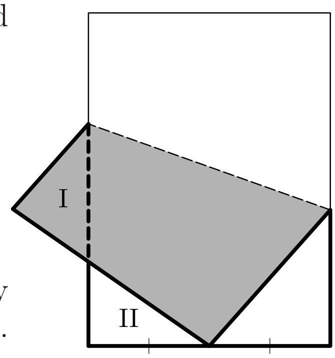
c) North will win.
d) North won't lose.
e) Three goals were made, in total.

After the match, it turned out that exactly three of these predictions turned out to be true. What was the final score?



4. A rectangular sheet of paper is folded so that one corner bisects the lower side, as shown in the figure above. Triangles I and II are congruent. If the length of the short side of the sheet of paper is 8, find the length of the long side.

5. The book *Magic for Dummies* contains the following passage:

"If you replace each of the distinct letters in GLOBALHELLFRY with distinct digits, and you get a prime number, then the world will suffer a terrible heat wave."

Is it possible to use this to actually create a heat wave?

6. The numbers from 1 to 16 are placed in a 4×4 table. Write down the largest number contained in each row, column, and diagonal (including diagonals containing just one cell of the table). The same number may be written more than once. Is it possible to write down

a) all 16 numbers, except, perhaps, two?
b) all 16 numbers, except, perhaps, one?
c) all 16 numbers?

Year 2010

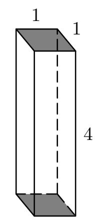
6th Class

1. The casing of a cylindrical sausage is marked with thin transverse rings. If it is cut along the red rings, you get 5 pieces. Cutting along the yellow rings produces 7 pieces, and cutting along the green rings, produces 11 pieces. How many pieces are obtained from this sausage if you cut all these colored rings?

2. In an enchanted forest there live only elves and gnomes. The gnomes lie when speaking about gold, but otherwise tell the truth. The elves tell the truth, except when speaking about gnomes, in which case they always lie. One day two inhabitants of this forest said the following:

A: I stole all of my gold from a dragon.
B: You're lying.

Is *A* an elf or gnome? How about *B*?



3. Piglet figured out how to build a parellelepiped (box) with unit cubes, and then completely cover it with three squares, with no gaps or overlaps. How did he do it?

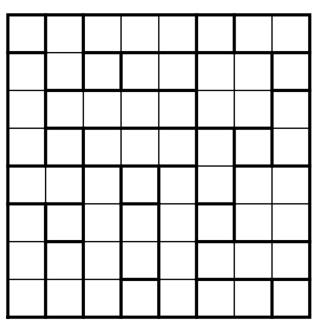
4. A currency exchange office conducts two types of trades:

a) In exchange for 2 euros, you get 3 dollars plus a complimentary piece of candy.

b) In exchange for 5 dollars, you receive 3 euros plus a complimentary piece of candy.

When Barney enters the exchange office, he has only dollars. When he leaves, he has fewer dollars, no euros, and 50 pieces of candy. How many dollars did he spend on these "complimentary" candies?

5. Sasha partitioned an 8×8 chessboard along the boundary lines of the squares to form 30 rectangles situated so that no two congruent rectangles are adjacent, even at corners, as in the figure.



Try to improve this result, forming more than 30 rectangles which satisfy these conditions.

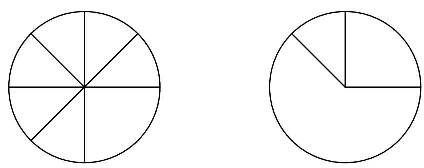
6. Thirty cups of tea were spaced at equal intervals along the edge of a round turntable. The White Rabbit and Alice sat at the turntable and begin drinking tea from two cups (not necessarily adjacent). When they finished their cups, Rabbit spun the table so that there were full cups in front of them. When they finished drinking these new cups, Rabbit repeated the process (possibly spinning the table a different amount), again so that there were full cups in front of them. This process continued until all the tea was consumed.

Show that had Rabbit always spun the table so that his new cup was one away from his previous cup (i.e., he drinks out of every other cup, consecutively), then again all the tea would have been consumed (i.e., at each stage, both of them would have full cups in front of them).

7th Class

1. Yuri has a calculator that can multiply a number by 3, add 3 to a number, or (if the number is divisible by 3) divide a number by 3. Starting with a display of "1," how can he get the value "11?"

2. Several wheels with spokes are on a vertical spindle. A view from above is shown in the left figure. Then the wheels turn, and the new view (from above) is as shown in the right figure.



Could there have been a) three wheels? b) two wheels?

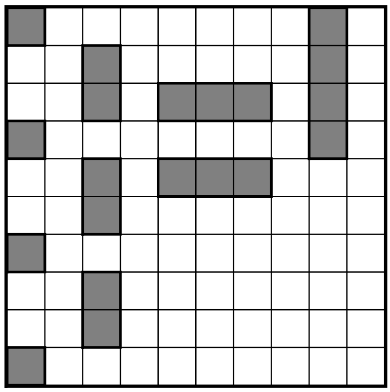
3. Several children ate some candy. Each child ate 7 pieces fewer than all the other children ate together, and each ate more than one piece of candy. How many pieces were eaten in all?

4. Rooster, Raven, and Cuckoo took part in a singing contest. Each judge voted for one of the three participants. Woodpecker counted 59 judges, with 15 judges voting for Rooster or Raven, 18 voting for Raven or Cuckoo, and 20 voting for Cuckoo or Rooster. Woodpecker counts poorly; however, each of the four numbers he counted above differ from reality by no more than 13. How many judges voted for Raven?

5. (a) Piglet figured out how to make a box out of unit cubes and cover it with three squares without gaps or overlaps. How did he do it? (b)

Can Piglet accomplish this in such a way that every pair of squares borders one another?

6. You have a certain fleet of Battleship-style ships, which can easily be deployed on a 10×10 board, as shown in the figure. What is the smallest square board that can hold this fleet, if no two boats are allowed to touch, even at the corners?

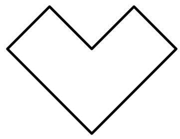


Year 2009

6th Class

1. The year 2009 has the following property: no matter how one rearranges its digits, one cannot obtain a smaller 4-digit number (we don't allow numbers to start with zeros). What is the next year that has this property?

2. Tile the following figure with 8 congruent pieces.



3. A park contains linden and maple trees. At the start of the year, 60% of the trees are maples, but some linden trees are planted in the spring, and then just 20% of the trees were maples. In the fall maples were planted, and then the percentage of maples was again 60%. By what factor did the population of trees grow during this year?

4. If an octopus has an even number of legs, it always tells the truth, but if it has an odd number of legs, it always lies. One day a green octopus said to a blue one, "I have eight legs, and you only have six."

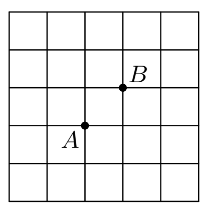
"I have *eight* legs, and you have just seven," indignantly replied the blue octopus.

"The blue octopus really does have eight legs," agreed the purple octopus, who boasted, "And I have nine!"

A striped octopus joined in, saying, "None of you have eight legs. I'm the only one who does!"

Which of these octopuses actually has eight legs?

5. A tourist wants to walk along the streets of Old Town from the train station (point A on the map) to his hotel (point B). He wants the route to be as long as possible, but without visiting the same intersection twice, since this is boring. Draw a longest such route on the map, and show that no longer route is possible.



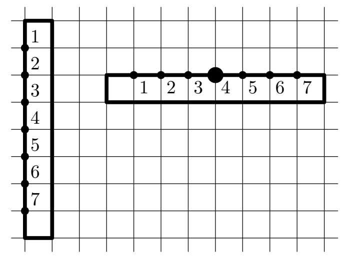
6. (a) A miserly nobleman stored gold coins in six treasure chests. One day, while counting them, he realized that if he opened any two chests, he could divide the coins in them so that each chest held the same number of coins. Then he realized that if he opened any three, four, or five chests, he also could distribute the coins so that each of the opened chests held the same number of coins. He was interrupted by a knock on the door, and the old miser

didn't have a chance to determine if he could divide his money equally among all six chests. Without examining the chests, is it possible to definitively answer this question?

(b) Suppose there were eight chests, and the nobleman was able to equally divide the money among any $2, 3, 4, 5, 6,$ or 7 opened chests. Can you definitively answer the question: Can he equally divide the coins contained in the eight chests?

7th Class

1. Petya and Vasya live in adjacent buildings, with apartment doors shown by dots in the figure — Vasya's is the one running across. Vasya lives in apartment #4. Petya wants to get to Vasya's apartment by the shortest route — he cannot go through building walls, but otherwise the route does not have to stay on the grid lines. He finds out that it doesn't matter whether he goes right or left from his doorway. Find Petya's apartment.



2. Grandpa planted the same number of turnips in each of two gardens. When Granddaughter visits the garden, she pulls 1/3 of the turnips out. When Beetle visits, she pulls 1/7 of the turnips out, and when Mouse visits, she pulls just 1/12 of the turnips out. At the end of the week, the first garden had 7 turnips left, and the second had 4. Did Beetle visit the second garden?

3. Among Neptune's servants are octopuses with six, seven, and eight legs. Those with seven legs always lie, while those with six or eight legs always tell the truth.

Four octopuses meet. The blue one says, "Together we have 28 legs. "

"Together we have 27 legs," says the green one.

"Together we have 26 legs," says the yellow one.

"Together we have 25 legs," says the red one.

How many legs does each octopus have?

4. A miserly nobleman stores gold coins in 77 chests. One day, while counting them, he realized that if he opened any two chests, that he could redistribute the total so that both chests would have equal numbers of coins. Then he realized that if he opened any 3, or any 4, ..., or any 76 chests, then he could also redistribute the total number of coins in the opened chests so that each chest had equal numbers of coins. A

knock at the door interrupted him, and the old miser was unable to check if he could equally divide all of his coins among his 77 chests. Is it possible to definitely determine whether this can be done, without looking in each chest?

5. On a sheet of graph paper, draw two quadrilaterals whose vertices lie on the gridpoints, so that you can join the two quadrilaterals to form a) a triangle, and (when joined in a different way) a pentagon; b) a triangle, a quadrilateral, and a pentagon (when joined in different ways).

6. Using for numbers as many coins as you want of $1, 2, 5$ and 10 ruble denominations, and using (for free) the four arithmetical operation symbols and parentheses, construct an expression that equals 2009 and costs the least possible amount of money.

Year 2008

6th Class

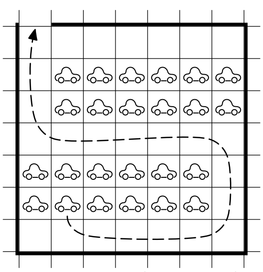
1. Today is 02/17/2008. Natasha noticed that the sum of the first four digits in this way of writing the date is equal to the sum of the last four digits. When will this coincidence happen for the last time in that same year?

2. A mother hare bought seven drums of different sizes for her seven young hares, and seven pairs of drumsticks of different lengths to go with them. If a young hare can see that both his drum and his drumsticks are bigger than those of at least one of his brothers then he starts drumming loudly. What is the greatest number of hares that can start drumming?

3. There were some whole blocks of cheese in a store. At night, some rats came along and ate 10 whole blocks, with each one of them eating an equal amount. Some of the rats got stomach aches from overeating. The remaining 7 rats finished the remaining cheese the following night, but this time each rat could only eat half the amount of cheese that they ate the previous night. How many blocks of cheese were there at the store to begin with?

4. Divide a square into squares of two different sizes so that the number of large and small squares is the same.

5. The parking lot in Flower Town is a square consisting of 7×7 small square spaces. A car can be parked in each of these spaces. The parking lot is surrounded by a fence, but one side of a corner square has been removed (that's the gate). A car moves along a one-square wide path. Dunno was asked to place as many cars as possible in the parking lot while making sure it is possible for every car to leave the parking lot at any time while the other cars remain stationary. Dunno placed 24 cars in the parking lot as shown in the picture. Arrange the cars in a different way in order to fit more cars in.



6. A beautiful Princess decided to lock an evil Dragon in a suite of four rooms, arranged north-south in a row and connected each to the next by a passageway. In each passageway, leaning against either the east or the west all, stands a large but weary guard. Each time the Dragon moves between neighboring rooms, the guard in the passageway crossed by the dragon switches to the opposite wall of the passageway and leans on it.

If all three guards ever lean against the east walls of their respective passageways, or all against the west walls, their weight will be too much and the walls will collapse, allowing the Dragon to escape. Can the Princess initially position the guards and the Dragon in such a way that he will never be able to escape?

7th Class

1. A certain number was multiplied by the sum of its digits, and the result was 2008. Find the original number.

2. Pump, Filter, Tap and Valve were the four teams participating in Waterway League Football Championship. Each team played against every other team once, with 3 points given for a victory, 1 for a draw, and 0 for a loss. Pump got the most points and team Valve got the least. Is it possible that Pump got only two more points than Valve?

3. Tim lives in a nine-floor apartment building. It takes him 1 minute to come down to the ground floor from his floor in the elevator. Tim cannot reach the button for his floor in the elevator because he isn't tall enough. So, to get up to his floor, he presses the highest button that he can reach and walks up the rest of the way. All in all, going up to his apartment takes 1 minute 10 seconds. The elevator moves up and down with the same speed and Tim climbs at half the speed of the elevator. Which floor does Tim live on?

4. See problem 6 for the 6th class.

5. Sergio cut two identical shapes out of cardboard and placed them, overlapping, at the bottom of a rectangular box. The cutouts overlapped, and together they fully covered the bottom of the box. A nail was then hammered into the center of the bottom. Is it possible that the nail pierced one of the cardboard shapes, but not the other?

6. While Basil was standing at a bus stop, one bus and two streetcars passed by. Some time later, Spy came to the same bus stop to keep a watch on someone. While he sat there, 10 buses came by. What is the minimum number of streetcars that could have passed during Spy's stay? Buses run around the clock at equal intervals of one hour, and streetcars also run around the clock at equal intervals.

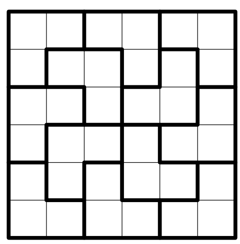
Solutions

Year 2011

6th Class

1. Two-thirds of the girls peeked with their right eye, while two-thirds of the boys peeked with their left. In all, two-thirds of the students had an eye that was not closed; thus 22 students peeked.

2. See the figure. The solution is unique, up to symmetry.



3. Since 3 predictions were true, then 2 were false.

First we will determine which team won. Suppose that North won. Then 4 predictions — (a), (b), (c), (d) — would be true, which contractions the fact that exactly 3 were true.

Suppose that the game ended in a draw. This makes predictions (a), (c), and (e) false (since in a draw, the total number of goals scored must be even). Thus, at most 2 predictions are true, again a contradiction.

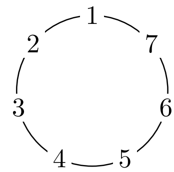
Therefore North lost the game. Then (c) and (d) must be false. This means that the remaining predictions must be true, namely that the game did not end in a draw, North scored against South, and there were exactly 3 goals total. Then South scored 2 goals, and North scored 1.

4. Since $8 \cdot HE < 100$, then $HE \leq 12$, and thus $H = 1$ and E equals 0 or 2. But E cannot equal 0, for then I and S would represent the same digit. Thus $E = 2$, and $8 \cdot HE = 96$, which means that US equals 96, 97, or 98. The last two cases are impossible, since they would force I

to equal 1 or 2, and these digits are already used. Thus $US = 96$, and $I = 0$.

5. Each gnome sees all but two caps: his own, and the hidden one. We need to determine how to find these two colors. Here is one approach.

Number the colors from 1 to 7 (for example, according to their order in the rainbow), and then arrange them in a circle, as shown on the right.



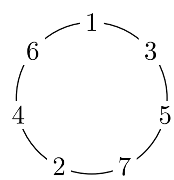
Each gnome will determine the two colors that he doesn't see, and name the one whose distance to the other is smaller when moving counterclockwise. Then three gnomes will guess correctly, and three will not. For example, if the hidden color is 1, then the gnomes wearing colors 2, 3, or 4 will guess it correctly.

There are other possible strategies. If a gnome doesn't see two colors of the same parity, then he names the one with the higher value, and if the two unseen colors have different parity, he names the smaller value. Exactly three gnomes will give the correct answer. For example, if the hidden color is 3, then it will be guessed correctly by the gnomes who wear colors 1, 4, and 6. The other cases can be argued analogously.

Remarks.

(1) The strategy used in the second solution can be applied to a round-robin chess tournament with an odd number of participants in which each person plays white and black against the same number of opponents. More precisely, each player gets a number, and if he meets a player with the same parity, then the one with the larger number plays white, while if the numbers are of different parity, then the lower number plays white. One can see that in this situation each player plays an equal number of games with the white pieces as with the black pieces.

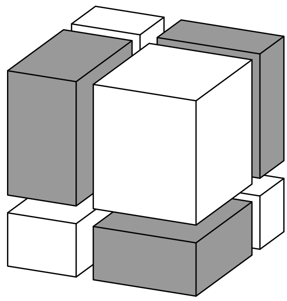
(2) Strictly speaking, the second solution differs from the first only by the numbering of the colors. If we place the colors on a circle like this:



then the strategy used in the first solution will now be precisely the one described by the second strategy.

(3) It can be shown that no strategy can result in more than half of the gnomes guessing correctly.

6. For each of the small blocks, the surface area made by the cuts is half the total surface area. Consider only the cut surfaces. Paint each small block black and white as follows (the hidden one is black):



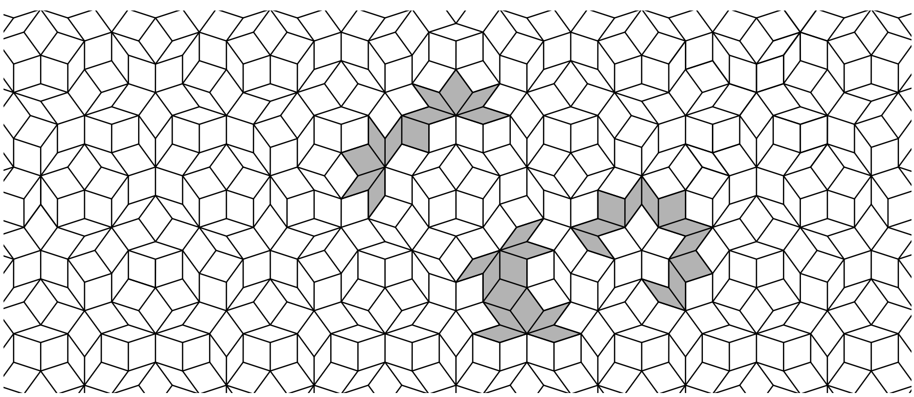
Then the cut surfaces come in pairs of surfaces that touch which are differently-colored, congruent rectangles.

Thus the sum of the areas of the black cuts must equal the sum of the areas of the white cuts. Then the sum of the surface area of the white blocks is equal to the sum of the surface area of the black ones. Consequently the surface area of the eighth block is

$$(148 + 46 + 72 + 28) - (88 + 126 + 58) = 22.$$

7th Class

1. Here are several solutions:

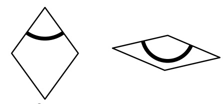


Remarks.

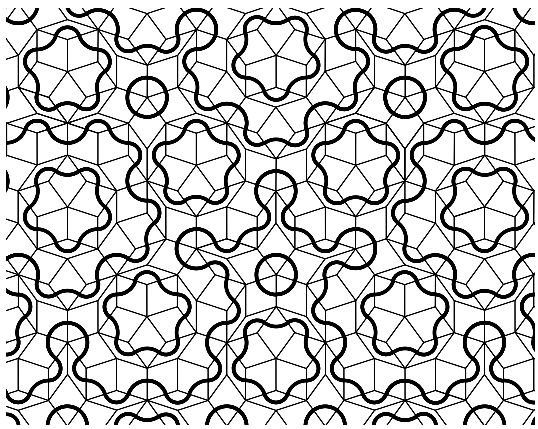
(1) Here's how we can find such solutions. Our figure must use many more skinny rhombuses than wide ones, yet there are clearly more wide rhombuses in the picture.

It is easy to see that each skinny rhombus borders at most one other skinny rhombus, and each wide one borders no more than two skinny ones. In order to put eight skinny rhombuses (four pairs) together into one polygon we need to include no fewer than three wide ones.

Now take a region of the mosaic, shade in all the pairs of bordering skinny rhombuses, and starting with some pair, keep trying to add a wide rhombus that connects with another pair of skinny ones, etc.

(2) Draw arcs on the rhombuses, like this: 

A *Penrose Tiling* describes a mosaic of these rhombuses where each arc is connected with the arc of a neighboring rhombus:



Our problem uses a Penrose tiling.

(3) If you draw (a portion of) a Penrose tiling on a (large) sheet of paper, and count the rhombuses, then the wide rhombuses will be approximately 1.6 times more numerous than the skinny ones. (In fact this ratio will approximately equal the *golden ratio*, getting closer and closer to it as the size of our paper grows.)

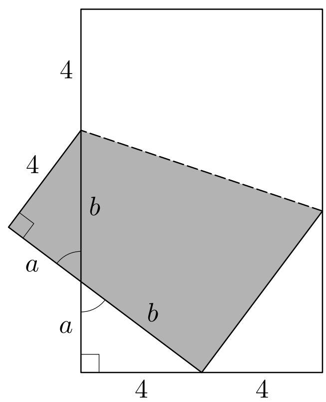
2. Since 10 flowers remained unwatered, $30 - 10 = 20$ flowers were watered. Consider the last tulip that Marge watered. Since there were 15 tulips, then 5 tulips remain unwatered.

Thus Homer picked these 5 tulips and stopped picking as soon as he reached the last tulip that Marge watered. This means that the rest of the watered flowers remain. In other words, $20 - 1 = 19$ remain.

3. See the solution to problem 3 for the 6th class.

4. Label the equal segments (figure below) using the fact that corresponding sides of congruent triangles are equal. We see that the length of the long side is equal to $a + b + 4$, and the length of the short side

is $a+b$. Thus $a+b=8$ and the long side is $a+b+4=8+4=12$.



Remark. Using the Pythagorean theorem, we can find the lengths of the sides of triangles I and II. This turns out to be the *Egyptian* triangle, with sides of length $3, 4$, and 5.

5. Counting the letters in GLOBALHELLFRY, we see that L occurs 4 times, while the other 9 letters each occur once. This means that if we replace the letters with digits, all the digits will appear exactly once, except for one: the digit that represents L appears four times.

The sum of the ten digits from 0 to 9 is 45, which is a multiple of 3. The sum of any three identical digits is also a multiple of 3. Thus, no matter how we replace the letters with digits, the sum of the digits will be a multiple of 3. This means that the number we get will be a multiple of 3.

The only prime that is a multiple of 3 is 3 itself, and since the number we are looking at is much bigger, it cannot be prime.

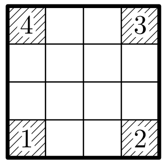
6. Observe that the numbers in the corners of the tables will be written down, since they are in diagonals of length 1. Of the remaining numbers, at least one will not be written. In fact, consider the smallest number that is not in one of the corners. It won't be written down, since every row, column, or diagonal containing it also contains other non-corner numbers. Thus, it is not possible to write down all 16 numbers.

Here is an example of a table where all but one of the 16 numbers will be written down:

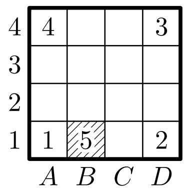
4	10	9	3
11	16	15	8
12	13	14	7
1	5	6	2

Remark. We can find this example by the following method. We will place the numbers one-by-one in order of size.

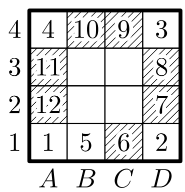
In order for 1 to be written, it must be in a diagonal of length 1, hence placed in a corner. Likewise, $2, 3, 4$ must be placed in corners, in any order:



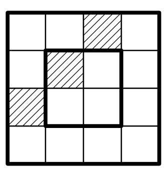
Now, wherever we place 5, we know that it won't be written (by the argument above). For example, we will put it in cell B1:



For 6 to be written, it must be placed on a row, column, or diagonal containing 5; for example, in cell C1. Next, 7 must be placed on a row, column, or diagonal containing 5 or 6, etc. This can be continued until we get the numbers up to 12 in a ring bordering the table:



The remaining four largest numbers can be placed in any of the four central cells, since each number in these cells is the largest value contained in a diagonal of length 3, as shown in the figure below.



Year 2010

6th Class

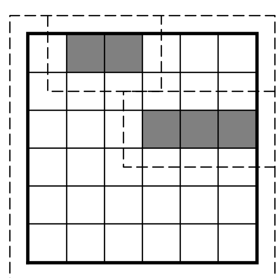
1. The number of pieces is always one more than the number of cuts. Hence, there are 4 red rings, 6 yellow rings, and 10 green rings. Thus there are $4+6+10=20$ rings in total, hence 21 pieces result.

2. Suppose that A were an elf. Then he told the truth, so B must be lying. But B is neither speaking about gold, nor about a gnome. So A must be a gnome, speaking about gold and lying. Thus B is telling the truth, about A, who is a gnome. Thus B cannot be an elf, and must be a gnome.

4. Since he received 50 pieces of candy, he conducted exactly 50 trades. During the course of these trades, he had to have converted all euros received into dollars. Thus, for every 3 trades of type (a), he had to have made 2 trades of type (b). Thus Barney received 3 dollars 30 times and gave away 5 dollars 20 times. His net expenditure was $20 \cdot 5 - 30 \cdot 3 = 10$ dollars.

5. *Partial solution.* As mentioned in the answer, we can show that the largest possible number of rectangles is 36. Our reasoning is similar to that used in the solution to Problem 6 for the 7th class. The demonstration begins by estimating the number of rectangles of different shapes.

- Unit rectangles: *There are no more than sixteen* 1×1 *rectangles.* If we partition the board into sixteen 2×2 squares, there can be at most one 1×1 rectangle in each of these squares.
- 2×1 rectangles: *No more than thirteen of these can be placed on the board.* Suppose there were N such rectangles. For the proof, extend the board with a half-unit wide boundary, and place such boundaries around each 2×1 rectangle:



The area of a 2×1 rectangle together with the boundary (called in what follows an extended rectangle) is exactly 6 units. Clearly these extended rectangles lie within the extended chessboard, and do not overlap one another. This means that the total area, which equals $6N$, cannot exceed the total area of the extended board (81), and hence $N \leq 13$.

- Other rectangles: Now we estimate the number of rectangles whose area exceeds 2.

  Suppose that there are x unit rectangles and y 2×1 rectangles. Then the number of remaining rectangles, each with area at least 3, is at most $(64 - x - 2y)/3$. This means that the total number of rectangles is at most

$$\frac{64 - x - 2y}{3} + x + y = \frac{64 + 2x + y}{3} \leq \frac{64 + 2 \cdot 16 + 13}{3} = 36\tfrac{1}{3} < 37.$$

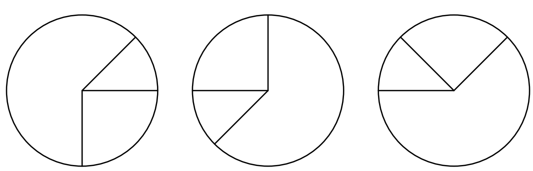
Unfortunately, these estimates do not preclude the possibility of 36 rectangles, and indeed there are even two sets of rectangles which appear to satisfy the estimates. However, a finer analysis, examining the relative positions of the rectangles on the board will show that these 36-rectangle sets are not possible.

6. Color the cups blue and red, alternating colors. Suppose Rabbit begins drinking from a red cup. We will show that Alice began with a blue cup. In fact, if she had drank from a red cup, then after each spin, two cups of the same color would be emptied. Since there are 15 cups of each color, at the end we'd have two cups of different colors, and no rotation of the table can place both of them in front of Alice and Rabbit simultaneously.

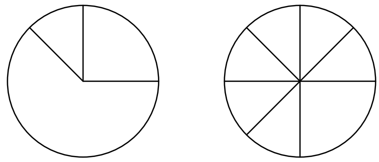
Now it is clear why Rabbit can always spin the table so that he drinks out of every other cup, consecutively: he will always have a red cup in front of him, and Alice will always drink from a blue one.

7th Class

2. This figure illustrates the solution to part a):



For part b), it is clear from the first figure below that no wheel has more than 3 spokes. The second figure shows that the total number of spokes is at least 7. Since $3 \cdot 2 = 6 < 7$, two wheels will not suffice.



3. Consider one child — Petya, say. If you took 7 away from the remaining pieces of candy, what is left will be precisely Petya's share. Thus, twice Petya's share must be equal to the total amount of candy, minus 7. This holds for all the children, so each child must have the same number of pieces of candy. Let's call this a "pile."

Clearly, each child ate a whole number of piles fewer than the total of what the remaining children ate, so the size of a single pile must divide evenly into 7. Since the size must be greater than 1, it must be equal to 7, i.e., each child eats one pile less than what the remaining children ate. Petya consumed one pile, and thus, the remaining children consumed two piles. The total amount of candy consumed is 3 piles, or 21.

We can also solve the problem algebraically.

Let S denote the total amount of candy that was eaten. If one child ate a pieces, then the rest of the children together ate $a+7$ pieces, and thus together $S = a + (a + 7) = 2a + 7$. This holds for each child, so each child consumes $a = (S - 7)/2$ pieces.

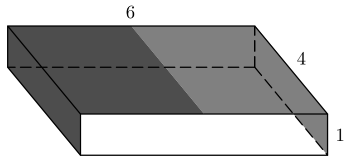
Now let N denote the number of children. Then the given conditions are equivalent to $a = a(N - 1) - 7$, which yields $7 = a(N - 2)$. Because 7 is prime, one of the factors in this expression is 1 and the other is 7. But it is given that $a > 1$, so $a = 7$ and $N - 2 = 1$. Thus $N = 3$, and $S = aN = 21$ pieces were consumed.

4. The number of votes for Rooster or Raven cannot be great than $15 + 13 = 28$. Likewise, the total number of votes for Raven or Cuckoo is at most $18 + 13 = 31$ and the number of votes for Cuckoo or Rooster is at most $20 + 13 = 33$. Adding these three totals gives us double the total number of votes (since each animal is counted twice). Thus, the total number of votes cast is no more than $(28 + 31 + 33)/2 = 46$. On the other hand, Woodpecker's first observation means that the number of judges cannot be less than $59 - 13 = 46$. Therefore there are 46 judges, as all of these inequalities have been turned into equalities.

Finally, the number of votes for Raven is found by subtracting the total number of judges from the sum of the votes for Cuckoo and Rooster: $46 - 33 = 13$ votes.

5. For part a), see the answer to problem 3 for the 6th class.

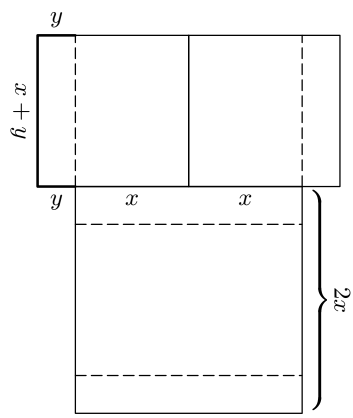
For part b), here is how one can accomplish this by covering a $1 \times 4 \times 6$ box with two 4×4 squares and one 6×6 square. The large square covers three faces: the front, back and lower, and each of the smaller squares covers half of the top and one of the two sides.



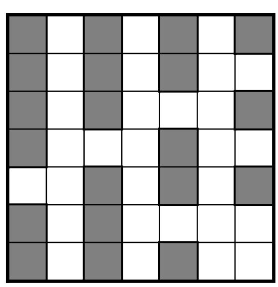
Remark. We can find dimensions of parallelepipeds and squares that work in the following way:

Draw a fold-out view of three squares, every pair of which borders the other (see the figure; the fold lines are dashed), and try to find the dimensions of the squares so that one can form a parallelepiped. Let the side length of the lower square be $2x$. The bottom side of one of the top squares is divided into segments of length y and x by the lower square.

When the parallelepiped is folded, the left side of the lower square will be glued to the thick lines. This yields the equation $y + (x + y) + y = 2x$, or $3y = x$. If we plug in $y = 1$, we get the answer given above.

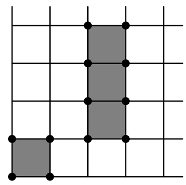


6. Here is a configuration of ships that works on a 7×7 board:



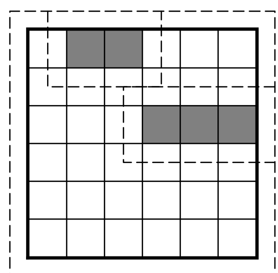
We now have to show that there is no solution on a 6×6 board.

First proof: For each ship, consider the points at the *corners* of occupied squares, rather than the unit square themselves. These corners are shown as dots in the partial board immediately below.



A 1×4 ship occupies $2 \cdot 5 = 10$ corners, and 1×3, 1×2, and 1×1 occupy $8, 6$, and 4, respectively. Since no corner can be occupied by more than one ship, the entire fleet must use $10 + 2 \cdot 8 + 3 \cdot 6 + 4 \cdot 4 = 60$ corners, and one cannot place it on a board with fewer corners. However, a 6×6 board has $7 \cdot 7 = 49$ corners.

We can reformulate this idea: cover each ship with a rectangle, and expand each rectangle by a half-unit on each side:



These rectangles cannot overlap and must collectively occupy $10 + 2 \cdot 8 + 3 \cdot 6 + 4 \cdot 4 = 60$ unit squares. But if we were able to place the ships on a 6×6 board, then the corresponding expanded rectangles would lie on a 7×7 board, which has only $49 < 60$ unit squares. (Note that each square of the new board contains exactly one corner from the old; thus the computation and conclusion are identical to the proof above.)

Second proof: Partition the 6×6 board into nine 2×2 squares. Each of these squares can only contain unit squares from at most one ship. Since there are 10 ships in the fleet, we cannot place them all on the board. (This argument also shows that no 10-ship fleet can be placed on a 6×6 board, even if all the ships were of size 1×1.)