Foreword

These days most everything we do in academia starts with an e-mail. It is no wonder then that something new and big was spawned on April 7, 2009, by an innocuous e-mail from a Kansas State University math professor: an e-mail so typical and so frequently coming to me, that I wouldn't have remembered it, except for its great consequences to the Berkeley Math Circle (BMC, `http://mathcircle.berkeley.edu`):

> My husband and I will visit MSRI in Fall 2009. We have a son who will be a 2nd grader at that time. We would like to ask you if there are any programs for this age in your math circle or in any other in the San Francisco Bay area? *Natasha Rozhkovskaya*

Our then-assistant, Ivan Matic (a Serbian graduate student at UC Berkeley), replied:

> I think that 2nd grade is too young for the math circle. We did have quite young kids, but as far as I remember the youngest we've had are 4th graders. *Ivan*

I could not have agreed more with Ivan's words: Come on, be serious! 2nd graders in the math circle?! Next there will be math circles for kindergartners!! And so I thought that this e-mail inquiry would die out, just like numerous previous parents' inquiries about meaningful and challenging math programs for such youngsters in the U.S. And yet, a surprising follow-up arrived:

> Then, if it turns out that there are no groups for this age, my complementary question would be, if you could kindly help me to create such a group for the fall semester. I lead myself a math circle for my son and his friends (there are 6 of us) in my town. So I can easily argue that they are too young for math—just the style of meetings is different. Since we will be only for one semester, the time to find other potential participants would be limited, I would greatly appreciate if you could help me to find other parents who could be interested in that. Thank you very much, for your time and help. *Natasha*

In the beginning, the idea of BMC-Elementary for 2nd graders sounded crazy to me. The math circles in Bulgaria (where I grew up) started in 4th grade, and I joined my own middle school math circle in 5th grade. After all, I founded the BMC itself in 1998 specifically for high school students, as a temporary project, to be continued in a year by teachers in their own schools in the San Francisco Bay Area. Still, what happened? We could not (figuratively or literally) kick the very dedicated and motivated 5th and 6th graders out of the BMC room: these were the siblings of the older high school circlers. At one point, a 3rd grade boy started accompanying his older brother, pulling a chair in the corridor, climbing on it and peeking through the room window into the circle session. And now look at the BMC, 14 years later: going stronger and stronger every year, bursting at the seams with over 230 students, and splitting into five levels of seven separate sections.

Indeed, BMC is NOT the temporary one-year project it was meant to be! As my (Russian) husband jokingly remarks: "There is nothing more permanent than the temporary!"

Considering the pioneering history of BMC, being overwhelmed by the hunger for better precollege math education in the U.S. at all levels and for all ages, and learning myself by-and-by about the great experience of teaching mathematics to very young kids in Russia, I was eventually conquered by the "madness" and gave the green light to a new-to-me and bold idea for the U.S.

Thus, in the fall of 2009, BMC-Elementary was born. Professor Natalia Rozhkovskaya, a mathematician visiting MSRI, championed the one-semester pilot project, envisioning it as a modest 12-to-15 student circle for 2nd graders.

We advertised the idea among the parents of the current BMC students, and before we knew it, there were 40 kids enrolled from grades 1–3. Quickly, two more Russian mathematicians were recruited: Laura Givental and Elena Blanter taught alongside Natasha that very first semester of BMC-Elementary. And so history repeated itself: once born, this level for young kids persevered—Elena and Laura, along with Professor Sergey Ovchinnikov from San Francisco State University, continued with dedication the "temporary pilot" project, now in its 3rd year.

Meanwhile, the 4th graders had fallen through the cracks: neither BMC-Elementary, nor BMC-Upper levels were designed for them. So, naturally, they came by the dozens—how could they not, when their older and younger siblings were already studying at BMC? Within a month in September 2009, I had to urgently add another BMC-Upper level, which I called BMC-Beginners, in order to absorb the incoming crowds of young budding mathematicians and separate them from the experienced international math olympians in the BMC-Advanced group. Once the ball starts rolling in the "right" direction, there is no stopping it. Two years later, in the fall of 2011, BMC has multiplied:

level	sections	
BMC-Elementary I	2	for 1st–2nd grades
BMC-Elementary II	2	for 3rd–4th grades
BMC-Beginners	1	for 5th–6th grades
BMC-Intermediate	1	for 7th–9th grades
BMC-Advanced	1	for 10th–12th grades

Each level also accepts a few exceptional cases of advanced younger students.

In the eyes of the Russian reader, there may be nothing astonishing in the story above. But think about it from the following viewpoint. While math circles in Russia and Eastern Europe have existed and blossomed for more than a century, drawing millions of youngsters and future famous mathematicians into mathematics and instilling in them life-long love for the subject, **there were no math circles in the U.S. until 1994**, when at last the Boston Math Circle opened up. Ironically, I was myself right at the place when this happened (doing my graduate work in mathematics at Harvard at the time), yet I did not find out about this circle until after founding the Berkeley Math Circle (BMC) in 1998. Thus, to my knowledge, the BMC is the second math circle in the U.S., followed in its birth only by a day by the San Jose Math Circle (SJMC, directed by Professor Tatiana Shubin of San Jose State University).

When BMC and SJMC started, the idea of math circles was brand new on the U.S. soil. It was almost unthinkable that a mathematician would dedicate time and energy to the math education of precollege kids, diverting precious time from his/her own research. At the same time, about a decade after the fall of the communist societies in Eastern Europe, enough immigrants from that region had come over to the U.S., only to face in person the gradual decay of U.S. public education, and to wish for a better education for their children. These immigrants dreamed of an education that approximated to some extent the former high-quality education they themselves had received back in their home countries.

To illustrate with an example, consider the latest no-homework-movement across the U.S. public school system. "What do you mean by 'no homework'?" my Bulgarian relatives would ask, raising their eyebrows in disbelief. The principal of my daughter's elementary school in Berkeley answers, "I am canceling and/or reducing homework for grades K–2 because some parents have complained that they do not get enough quality time with their kids." A math teacher at a workshop at MSRI whispers unhappily into my ear, "We are told by the school administration that some kids have parents to help them with homework, and some don't. And hence, they say, it is 'unfair' to give homework at all." There is no way to explain such reasoning other than holding U.S. education a hostage to U.S. social problems: instead of working with the kids who don't have parents to help them with the homework, the society deprives even those kids of any meaningful homework practices and simultaneously works to eliminate the historically first and foremost factor in a person's development—the family factor. Needless to say, such attempts at "social engineering" inevitably end up creating even deeper inequity among the young students: those who before had parents caring for them would now have parents passionate about giving them everything to compensate for their impoverished public education. One cannot eliminate the family as a crucial developmental factor in a child's life. I came to this understanding myself after a year of battling with the Berkeley public elementary school system, teaching my daughter every day with any and all Bulgarian and Russian textbooks our relatives could send overseas, and eventually transferring her to a private school nearby.

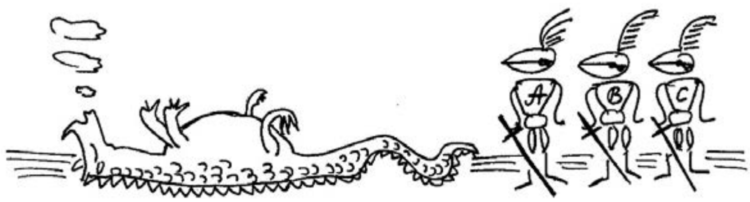
With this background information, perhaps, the reader from Russia would now see why the Berkeley Math Circle was started at the University of California at Berkeley, and why it has continued with such vigor to develop and grow over the last 14 years, now incorporating all grades from 1st to 12th. Sooner than I anticipated, my own daughter started to participate in BMC-Elementary as a kindergartner, and this very year will be her second year there. Sooner than we would have liked, we developed long waitlists caused by a shortage of rooms to accommodate all the students who desired to join BMC. Sooner than we all dreamt, we were surrounded by more than 100 new math circles across the U.S. and neighboring countries: all aspiring to pull the young generation into the mathematical heights that are still remembered by us, the recent immigrants, as once being the standards in Eastern European education. It is the hunger for excellence and joy of sharing the mathematical wonders with the young kids that have kept the Berkeley Math Circle fire alive for so long, through hardships and through successes.

And it is to the credit of Natalia Rozhkovskaya and her collaborators to bring to fruition the "crazy" idea of a *U.S. circle for 6–8 year olds.* Because all gold medals at the International Mathematical Olympiad, all first prizes at the Intel Science Talent Search, and all Nobel prizes or Fields medals start with a little kid wondering and discovering.

With gratitude and hope,
Zvezdelina Stankova
Berkeley Math Circle Director
Berkeley, CA,
August 16, 2011

Lesson 12

Knights and a dragon



Three knights came to their king and told him that they had killed a dragon.

Knight A: "Knight B killed the dragon!"
Knight B: "Knight C killed the dragon!"
Knight C: "I killed the dragon!"

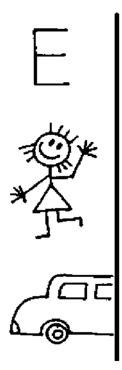
Problem 12.1. If all the knights lied, who must have killed the dragon?

Problem 12.2. If only one knight told the truth, then who killed the dragon?

Problem 12.3. If two knights told the truth and one lied, then who killed the dragon?

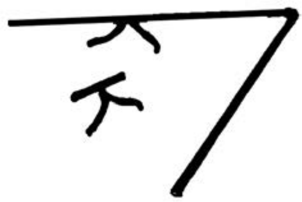
One mirror

Problem 12.4. The letter E came along and looked into a mirror. What did E's reflection look like? Draw it and reflections of other figures. (The vertical line in the picture represents the mirror.)

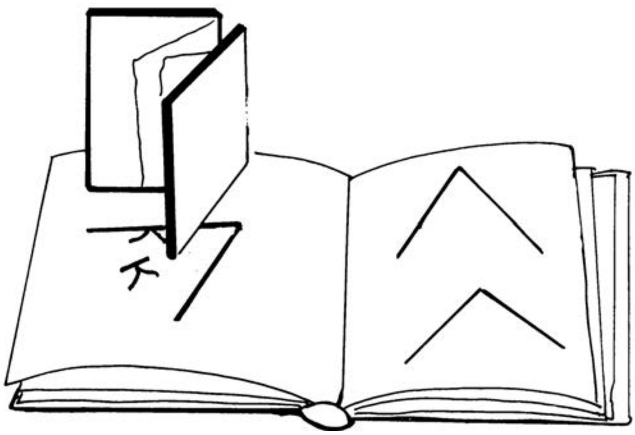


Two mirrors

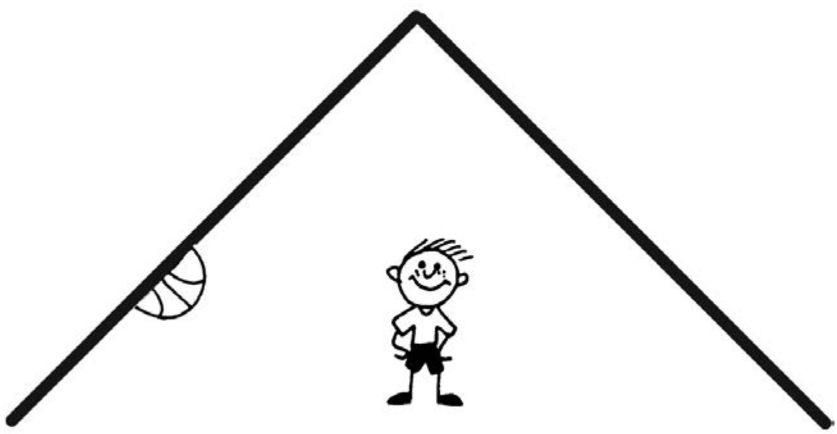
For the activities of this lesson you will need two pocket mirrors. Whenever you see a picture of an angle like this one:



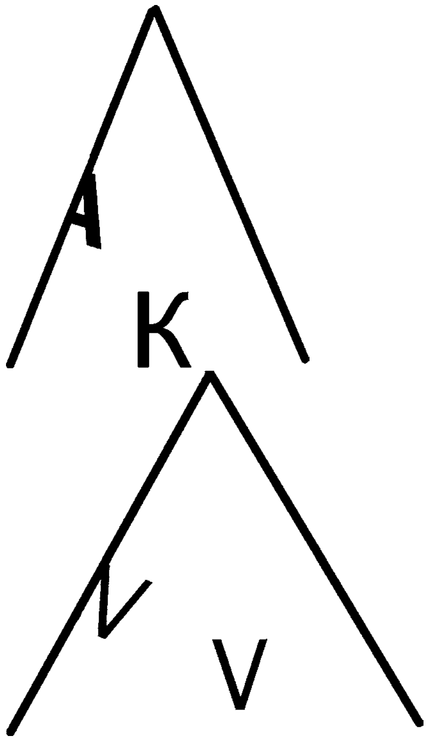
arrange your mirrors vertically along the sides of the angle to form a mirrored corner.



Problem 12.5. Put your mirrors on the sides of the angle and look into the corner of the mirrors. How many boys do you see? How many balls do you see?



Problem 12.6. What letters and how many of each do you see in the mirrors in this picture?

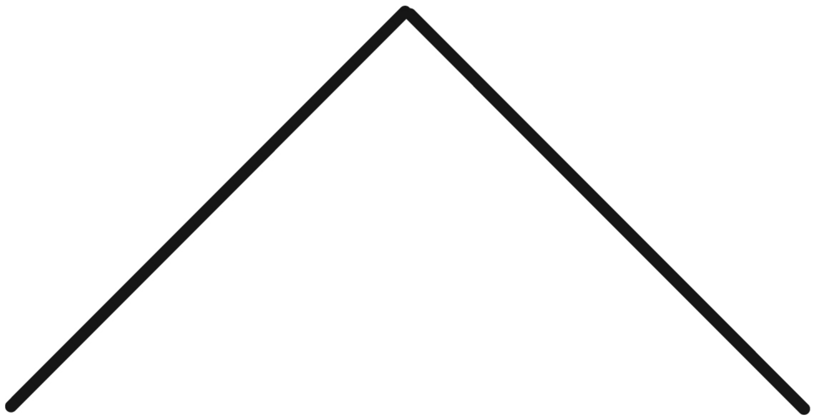


Problem 12.7.
1) Make a corner with your mirrors so that you can see six letter W's.
2) Make a corner with your mirrors so that you can see eight letter W's.

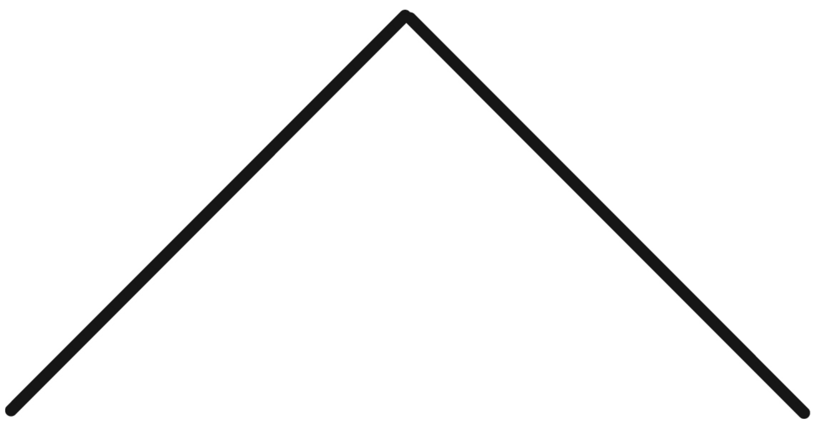
W

Problem 12.8.

1) Draw some boys and some balls inside this angle so that you see exactly four boys and one ball in your mirrors.



2) Draw some girls in this angle so that you see six girls in the mirrors.



Old lady in the mirrors

Problem 12.9.[8] Look at this picture of an old lady. Can you tell how many mirrors were used for this photo? What was the angle between the mirrors? Which lady is the real one?

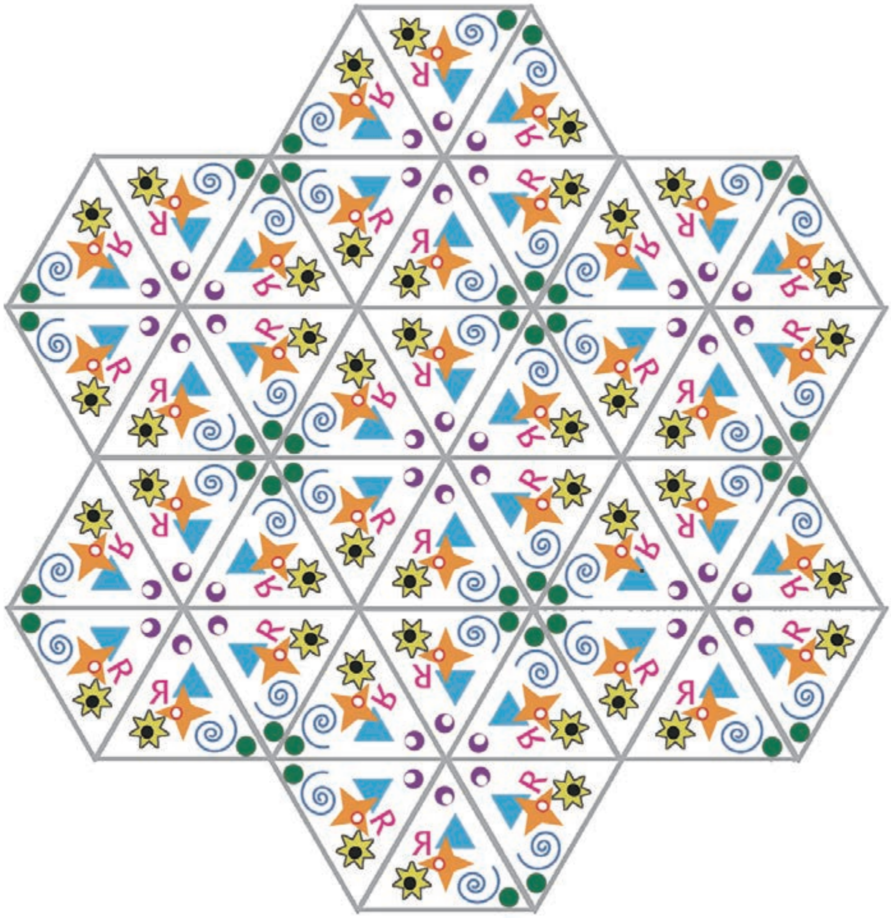
Kaleidoscope

Have you ever seen a kaleidoscope, and do you know how one works? Let's think about what we see when looking into a kaleidoscope and why the picture looks so beautiful.

The inside of a kaleidoscope is made of a tube of three long rectangular mirrors. The tube also contains colorful bits of glass or beads. When we look into a kaleidoscope, we see an infinite number of reflections of the beads on mirrors in the walls of the tube. This makes an infinite triangle lattice with a repeating symmetric pattern. The symmetry is what makes the picture beautiful.

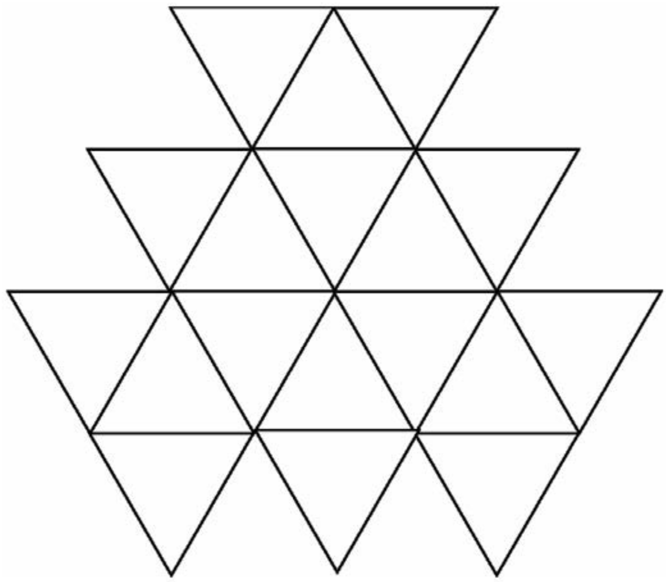
A kaleidoscope that contains the beads  makes the picture below.

[8]I am grateful to Professor Kevin Mitchell for kind permission to reproduce here this photo of an unknown photographer from his private collection.

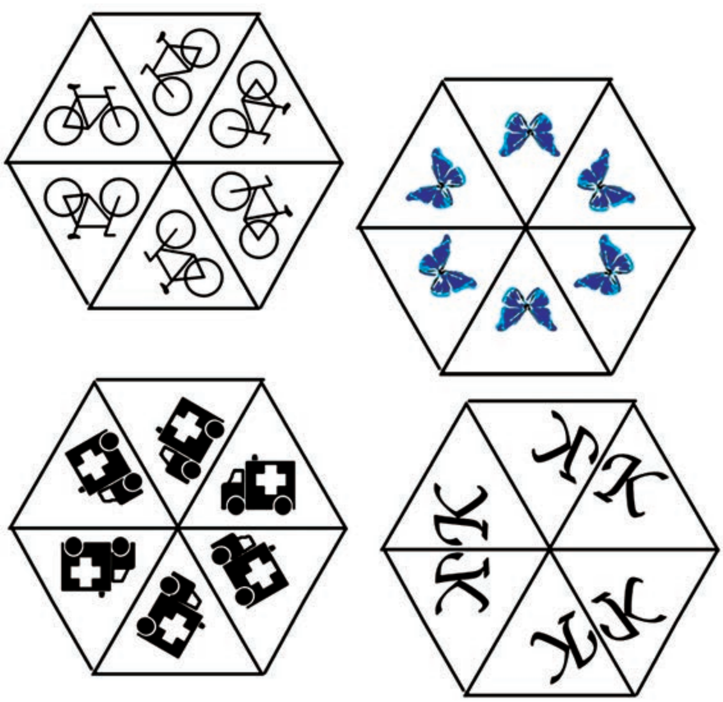


In many science and children's museums, visitors can get inside a huge kaleidoscope or even in a mirror labyrinth. The photo of a boy and his mom shows how it might look.

Problem 12.10. In the triangle net below, draw a pattern that one could see in a kaleidoscope.



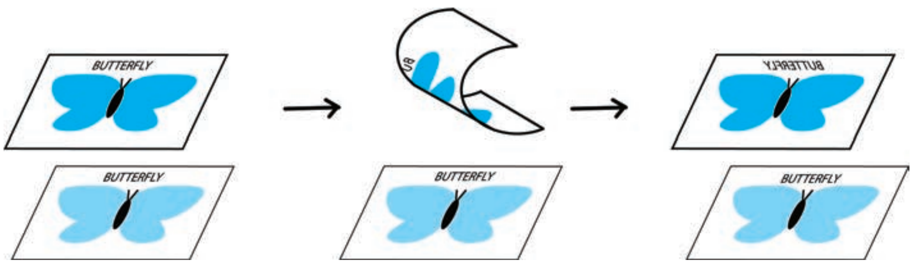
Problem 12.11. Which of these pictures could be a part of a pattern in a kaleidoscope, and which ones could not?



Symmetry

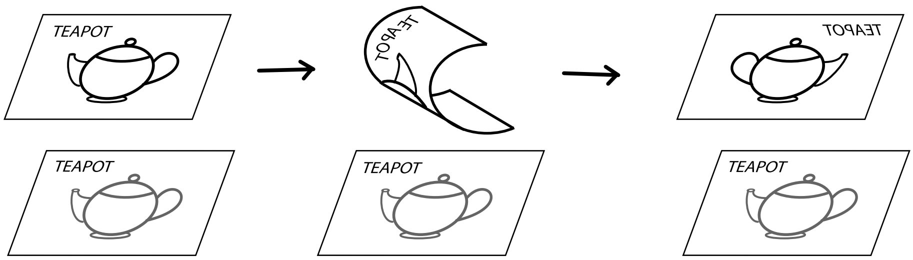
We observed that a picture in a kaleidoscope is symmetric, but what does the word "symmetric" mean? How do we determine that the butterfly and the star are symmetric, and that the tea pot is not?

Imagine that a butterfly is drawn on a piece of paper and above it there is an identical butterfly on a transparent sheet.



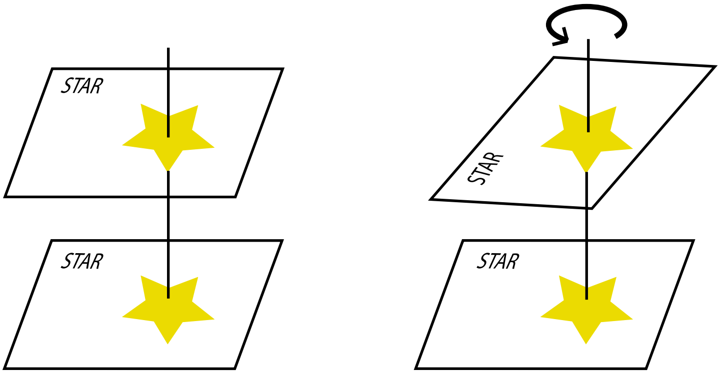
If we turn over the transparent sheet to the other side, we can still place the picture of the butterfly on the transparent sheet over the butterfly on the paper as before, and the pictures will match up exactly. This is due to the *reflection symmetry* of the butterfly.

What about the picture of the tea pot?



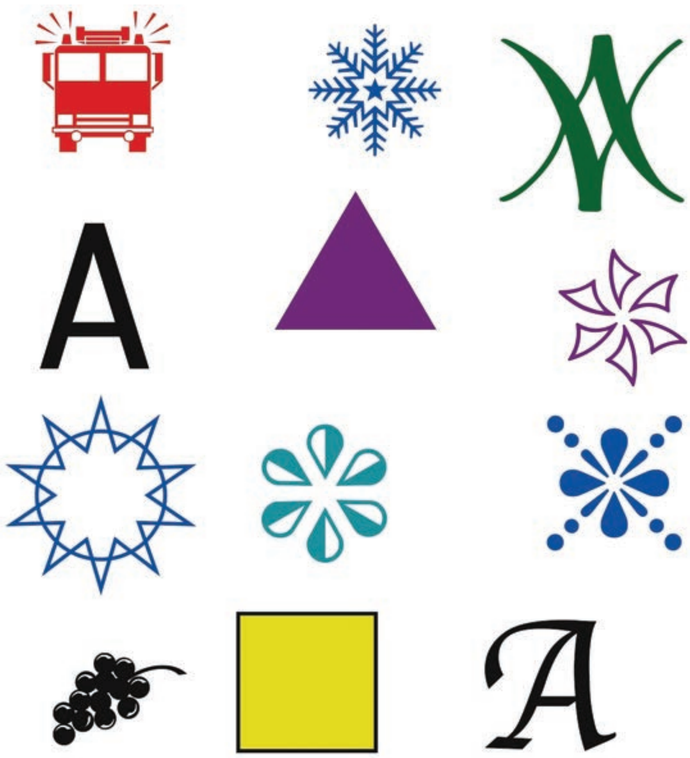
If we imagine the two sheets with the teapot instead of the butterfly and we turn over the transparent sheet, there is no way to match it with the picture on the paper: the spouts would point in different directions! The drawing of a tea pot does not have reflection symmetry.

Of course, the star does have a reflection symmetry, but not only that. The transparent sheet can be rotated around the center of the star so that the picture would again match the one on the paper. This is because the star has *rotational symmetry*.



These thought experiments can be summarized as follows. In every experiment we made some transformation of the plane (transparent sheet). It was a rotation (when we rotated the page around one point) or reflection (when we turned over the page). The picture of an object under the transformation did not change and this is what gives it symmetry. There are other types of symmetry in addition to reflection and rotational symmetry, but we will not discuss them here.

Problem 12.12. Which pictures below have reflection symmetry? Which pictures have rotational symmetry?



At the lesson

1. *Knights and a dragon.* This part of the lesson went very smoothly. To the great satisfaction of instructors, participants not only answered each question correctly, but they provided very clear and well-organized arguments.

2. *Mirrors.* We purchased several kaleidoscopes for the discussion of symmetry, but unfortunately that time I was not able to provide even one pair of pocket mirrors. Therefore, the problems were offered to students in a slightly different form from what is presented in this chapter. Students had to solve similar problems in their minds. This did not cause any difficulties in Problem 12.4, but imagining two mirrors was already challenging, and when we moved to a picture in a kaleidoscope, only very few participants were able to complete it. Later, at the Math Circle Seminar at Kansas State University, the tasks of this lesson were given exactly as they are described in this book. This time I was better prepared for the workshop, and everyone in class had a pair of small mirrors.

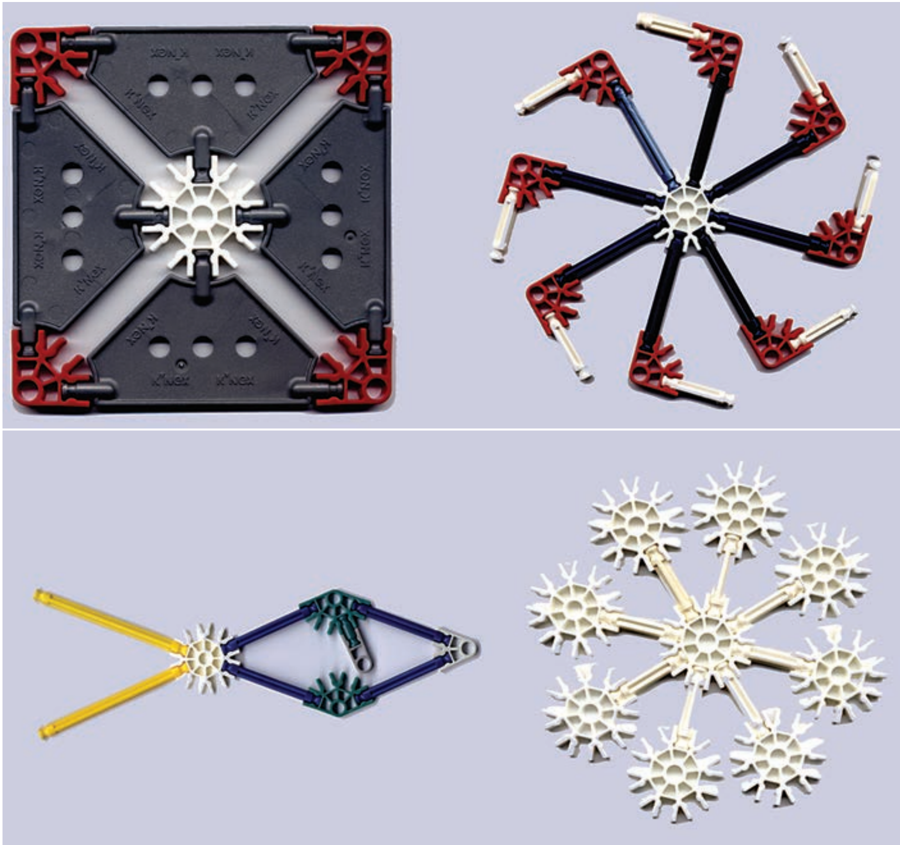
3. *Kaleidoscope.* All participants have seen kaleidoscopes before, but, to my greatest surprise, the following facts were revealed:

• Not everyone in class knew the name of the toy.

• Kids struggled to explain what they see in a kaleidoscope. It seems that many of them did not grasp the structure of the picture such as the triangle lattice and the symmetry. For example, "I see a cool picture" was an explanation given by one of the participants. Even those students who recognized symmetry could not find the words to describe it. One of the best descriptions that we could get was "There are many equal parts".

• Very few participants had ever thought about how a kaleidoscope works. Some of them knew that there are mirrors inside, but when I asked how many, the estimate varied from 1 to 18.

4. *Symmetry.* Anyone can recognize symmetry of an object, but few can give a definition of symmetry. In mathematics symmetry means an invariance of an object under some transformation. As examples of reflection and rotation I wanted to introduce symmetry exactly as a property of invariance. To illustrate this approach, we used figures made from the construction set K'nex. We put the figures on the board and asked students to show a figure that "did not belong to the sequence" and explain why. Immediately two candidates appeared: the wheel and the fish. The fish became the final choice for the object that did not belong to the sequence, since, using the words of one participant, "The wheel looks suspicious, but the fish looks even more suspicious." The discussion helped to find a reasonable argument: another participant nicely explained that the fish does not belong to the sequence since it does not have symmetry. Someone else added that if the eye was removed, the fish would become symmetric too. It is not surprising that there were doubts about the symmetry of the wheel: quite often in common language the word "symmetry" means exclusively reflection symmetry.

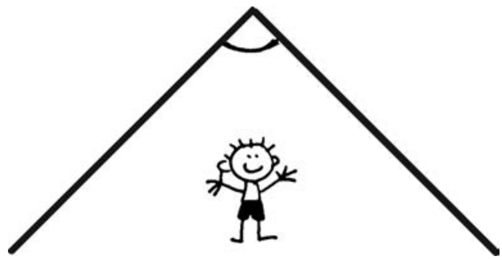


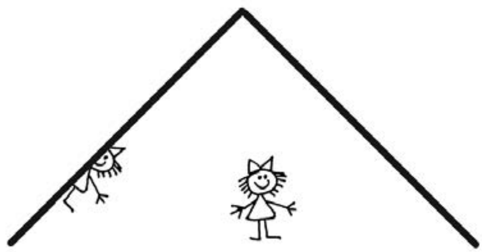
To illustrate symmetry as an invariant of a transformation, I used "shadows" of the figures made on a copy machine. We flipped and rotated the figures and noted whether or not they coincided with their "shadows" after these transformations.

We concluded the lesson with a classical activity on symmetry: paper snowflakes. The templates were taken from the website `http://www.KinderArt.com`.

Answers

12.1. A.
12.2. B.
12.3. C.
12.8.

1) 

2) 

12.9. Two mirrors were used for the photo and they were placed at an almost right angle to each other. Most likely, the lady that turned her back to us is a real one. Note that in the picture of a boy and his mother inside the kaleidoscope in a science museum there are no "real" people—what we see in the picture is the photo taken by the visitors—the reflection of themselves in one of the mirrors.

12.11. Bicycles and letters "K" can be part of a picture in a kaleidoscope, but butterflies and trucks cannot.

12.12. The next table gives the answers to this problem. The position in the picture of Problem 12.12 corresponds to the positions in the table.

reflection symmetry	reflection symmetry rotational symmetry	rotational symmetry
reflection symmetry	reflection symmetry rotational symmetry	rotational symmetry
reflection symmetry rotational symmetry	rotational symmetry	reflection symmetry rotational symmetry
	reflection symmetry rotational symmetry	

M.11. How old are you?

How old are you? Yes, this is an easy question. Of course you know the answer. You, probably, answered "I am so-many years old." But what do you mean by a "year"?

Problem 11.1. What period of time is called a "year"?

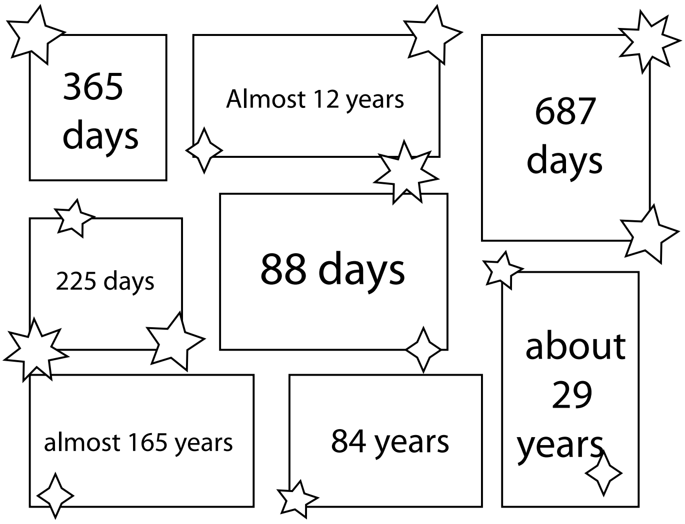
The answer to the previous question tells us that a year on a different planet could be longer or shorter than our Earth year.

Problem 11.2. In the first column of the table below, write all the planets of our solar system, starting from the one that is the closest to the sun and ending with the most distant planet.

Problem 11.3. Which of the planets in your list has the shortest year? Which one has the longest year?

No.	Planet	Length of a year (in Earth days or Earth years)
1		
2		
3		
4		
5		
6		
7		
8		

Problem 11.4. Below is the list of the lengths of years on all the planets in the solar system. The lengths are given in Earth days or Earth years. Match the lengths of the years with the names of planets and write them in the table.



Problem 11.5. One Earth year is slightly more than four Mercury years. How old would you be on Mercury?

Problem 11.6. One Earth year is about half of one Mars year. How old would you be on Mars?

Answers

11.1. A year is the period of time that is required for a given planet to go around the sun.

11.2. and **11.4**.

No.	Planet	Length of a year (in Earth days or Earth years)
1	Mercury	88 days
2	Venus	225 days
3	Earth	365 days
4	Mars	687 days
5	Jupiter	almost 12 years
6	Saturn	about 29 years
7	Uranus	about 84 years
8	Neptune	almost 165 years

11.3. Mercury has the shortest year; Neptune has the longest year.