Introduction

"The Berkeley Math Circle was really critical in my development. It was the best method available not only to get a flow of mathematical ideas and problems to think about each week but also to meet other interested students and professional mathematicians from all over the Bay Area. You get stimulation from exchanging ideas with other people that you don't get from reading books at home.

I can also testify to the usefulness of studying mathematics even for students who don't plan on doing it as a career. For someone who wants to go into, say, law, policy analysis, philosophy, economics, or computer science, the kind of logical, abstract thinking that mathematics develops is really the best preparation. I realize that the Circle is most interested in attracting students whose lifelong passion is for mathematics, but it also helps others along the way."

Gabriel Carroll, BMC alumnus
Perfect IMO '01 score
Four-time Putnam Fellow
Assistant Professor of Economics, Stanford

1. Top-Tier Math Circles[1]

This book is based on material from a dozen of the 800 sessions of the Berkeley Math Circle (BMC), held over the past 16 years. BMC has been described as a *top-tier math circle*, calling for the following two definitions.

1.1. Math circles are weekly math programs that attract elementary, middle, and high school students to mathematics by exposing them to intriguing and intellectually stimulating topics, rarely encountered in classrooms. Math circles vary in their organization, styles of sessions, and goals. But they all have one thing in common: *to inspire in students an understanding of and a lifelong love for mathematics.*

[1] Based on contributions from Marc Whitlow and Mike Breen (BMC Parents), Zvezdelina Stankova (BMC Director), and Tatiana Shubin (SJMC Director).

1.2. Top-tier math circles *prepare our best young minds for their future roles as mathematics leaders.* Sessions are taught by accomplished mathematicians and explore advanced mathematical areas. They provide an educational opportunity for top pre-college mathematics students, not offered in any other setting in the U.S. education system. In addition to learning advanced mathematics topics, students are taught the technical writing skills needed to convey the solutions of complex problems.

As an example of a top-tier math circle, the **Berkeley Math Circle** is fashioned after the leading models in Eastern Europe, where math circles originated over a century ago. BMC itself started in the fall of 1998 with about 50 students, primarily in grades 7-12, and there was only one session per week that lasted 2 hours. Sixteen years after, the circle has expanded to about 300 students in grades 1-12, split into two major groups:

- **BMC-Upper** with 3 levels: *BMC-Beginners* for 5^{th}-6^{th} grades (1.5 hours per week); *BMC-Intermediate* for 7^{th}-8^{th} grades (2 hours per week); and *BMC-Advanced* for 9^{th}-12^{th} grades (2 hours per week). BMC-Upper is directed by Zvezdelina Stankova.
- **BMC-Elementary** with 2 levels: *BMC-Elementary I* for 1^{st}-2^{nd} grades (3 sections, 1 hour per week); and *BMC-Elementary II* for 3^{rd}-4^{th} grades (3 sections, 1 hour per week). BMC-Elementary is directed by Laura Givental.

This book series is based on sessions from **BMC-Upper** and from the **original BMC**, when there was only one group for all. To save space, "BMC" throughout this book will refer, for the most part, to materials, instructors, and students from "**BMC-Upper**."

Like top-tier universities, BMC

- challenges students with beautiful, difficult mathematical theories,
- introduces them to powerful problem-solving techniques,
- constantly provokes deep thought, and
- inspires the creation of original ideas.

Topics covered at BMC include combinatorics, graph theory, linear algebra, geometric transformations, recursive sequences, series, set theory, group theory, number theory, elliptic curves, algebraic geometry, applications to computer science, natural sciences, economics,and many more. Each topic is taught by an *expert in the field* who has the ability to challenge the students and support them as they attempt to meet these challenges. All problems require students tocome up with *mathematical proofs*. Proofs put forward by the students are not always the most eloquent. Only an accomplished mathematician can understand where a student might be heading in his/her proof and offer assistance through this challenge.[2]

[2] For examples of noteworthy past and present instructors who have brought their world expertise to BMC, see the Epilogue.

The sessions are fast-paced and intellectually demanding. It is hard to convey just how advanced this subject matter is without actually attending a session; but comparable levels can be found in advanced undergraduate and beginning graduate courses.

The *Monthly Contests* (MC) at BMC can also convey the depth of the material. These are take-home exams of four or five hard, thought-provoking problems, requiring independent research, split into two levels: *MC-Beginners* (up to grade 8) and *MC-Advanced* (up to grade 12). In the beginning years of BMC, the monthly contests were designed and graded by UCB faculty. However, for the last 14 years the MC were designed and coordinated by current and former circlers.

The MC develop not only advanced understanding, but also *technical writing skills*: the students must describe on paper, convincingly and without gaps, how they solved a problem. This is a fundamental skill and key to making intellectual property contributions; it is a unique feature of the top-tier math circles, not found in middle or high schools, where students are taught to meet state standards on questions that take less than a minute to answer. In contrast, monthly contest problems may take the best students hours or days of concentrated thought. Only a few participants are capable of solving all the problems; yet, through the attempt everyone learns about *the real world of mathematical research.*

1.3. The next generation of math leaders. The students of BMC come from a variety of socio-economic and ethnic backgrounds. The proportion of female to male students is approximately 2:3. This is an amazingly high ratio considering the trend of other high-level math programs, which are "male-dominated" or "male-only." Excellent role models for the female students are provided by the female directors of the top-tier math circles in Berkeley [**11**], San Jose [**71**], Los Angeles [**47**], and (formerly of) Marin Math Circle [**52**]; but perhaps even more important to the students are the outstanding lectures given by dozens of *female* professors and graduate students.

Currently, BMC does not actively recruit participants. Students and their parents find out about the circles by word of mouth, from the Circle's web site, `http://mathcircle.berkeley.edu/`, through local universities, and in publications. Due to an increased number of applicants, there is a semi-formal selection process based on several open essay-type questions along the lines of:

- Describe your mathematical background and experiences so far.
- Why do you want to join BMC? What do you expect from BMC?
- What is your favorite math problem that you *can* solve? State and solve the problem. Why is it your favorite?
- What is your favorite math problem that you *cannot* solve? State the problem and explain why you cannot solve it but why you would like to solve it.

Needless to say, BMC students are usually years ahead of their peers: they often complete most of high school mathematics by age 13 (8^{th} grade), some take many college math major courses by the time they graduate from high school, and a few of the top circlers venture into graduate courses and serious mathematical research even before entering a university. The *accomplishments of students* who have benefited from BMC can be measured in many ways. For example, a number of these students have gone on to win International Math Olympiad medals and Putnam awards, and the majority have been admitted to top-tier universities. BMC and the other top-tier math circles not only produce highly accomplished students – they produce and train *the next generation of leaders in mathematics.*[3]

2. Why, What, and for Whom?

Running BMC for 16 years has taught us a lot about math education in the U.S. and has helped us to understand better our own childhood education and origins of our passion for mathematics. To share this experience with you, the reader, is the *purpose of this book*:

- to present you with beautiful theories, problem-solving techniques, and mathematical insights;
- to provide you with an abundance of exercises and problems to work on and with ready materials for math circle sessions.

2.1. The middle or high school student who is interested in expanding his/her math horizons and going well beyond anything that the regular math classroom can offer, who is brave enough to tackle non-trivial math ideas and work on hard problems for hours, who loves challenges and is motivated to overcome them: this is *the ideal reader of the book.*

Don't confuse the above description with "top" or "brilliant" students: you will never know if you are talented in math unless you give it a try. And you may be pleasantly surprised by what you find out: that mathematics is a whole lot more than "adding fractions," "algebraic manipulations," or "endless quadratic equations" in homework assignments. You will discover that Calculus is not the "pinnacle" of mathematical knowledge (as thought by many): it is only one of many beginnings, part of the subject of real analysis. Indeed, *other* wonderful topics are awaiting you (cf. Fig. 1, p. xviii):

- multiplicative functions in number theory;
- knot theory in topology;
- Rubik's Cube and groups in abstract algebra;
- interaction between geometry, trigonometry, physics, and Calculus;
- complex numbers arising from algebra and applied to geometry;
- game theory and inequalities attacked by monovariants; and
- plenty of proof methods and problem-solving techniques.

[3] To learn about the need for top-tier math circles, we direct the reader to the Epilogue.

2.2. Prerequisites. To read the book comfortably, you do *not* need to have *Calculus* under your belt, except

- in the very last section of Session 12 on plane geometry, which discusses a *series* solution to a geometric question, or
- if you want to *prove* the cited theorems in Session 9 on inequalities.

However, familiarity with basic geometry and algebra concepts and theorems will definitely be helpful; e.g., lines, circles, triangles, rectangles, trapezoids, and quadrilaterals in general; similarity criteria for triangles and the Pythagorean Theorem; equal alternate interior angles for parallel lines and bisecting diagonals in a parallelogram; integers, divisibility and remainders; operations on fractions and real numbers, intervals and sets of numbers; and manipulations of algebraic expressions written with letters. In some sessions, functions will play a major role; hence having studied some basic (pre-calculus) examples will not hurt; e.g., linear and quadratic functions, polynomials, exponential and trigonometric functions, as well as their graphs.

The above concepts will be re-introduced via examples in the book. But if you feel that you need more solid background, we direct you to several wonderful books that should be part of any budding mathematician's library:

- *Geometry, Book 1* by Kiselev [**32**],
- *Functions and Graphs* [**27**], *The Method of Coordinates* [**28**], *Sequences, Combinations, Limits* [**31**], *Algebra* [**30**] and *Trigonometry* [**29**] by Gelfand, et al.,
- for the older reader, *103 Trigonometry Problems from the Training of the USA IMO Team* by Andreescu and Feng [**5**].

2.3. The logical structure of the book series (volumes I and II) is outlined in Figure 2 on page xviii. A solid arrow indicates that a session requires its "predecessor" to be studied beforehand, while a dashed arrow indicates that the "predecessor" will be helpful but is not absolutely necessary. For example, in order to understand Rubik's Cube II, one should first study Rubik's Cube I; on the other hand, Rubik's Cube I-II will make Group Theory I more concrete, but they are certainly not mandatory.

Sessions that are bubbled in an ellipse can be attempted without any prerequisites, while sessions encompassed in a rectangle have at least one necessary predecessor. For example, Monovariants II calls for a prior study of Monovariants I, while Knot Theory can be attacked with little reference to other sessions. Sessions **not** enclosed in anything are from volume I.

Finally, there is a group of sessions that pertain to general proof methods, PSTs, and theory that appear in most other places. These sessions are from volume I and are roughly grouped in the two nebulous "clouds": Proofs I-II, PSTs, and Induction in one "cloud," and Number Theory I and Combinatorics I in another "cloud." Figure 2 captures some, but certainly not all, relations among the sessions and topics. The reader is welcome to search for and draw more arrows, as he/she goes through the book.

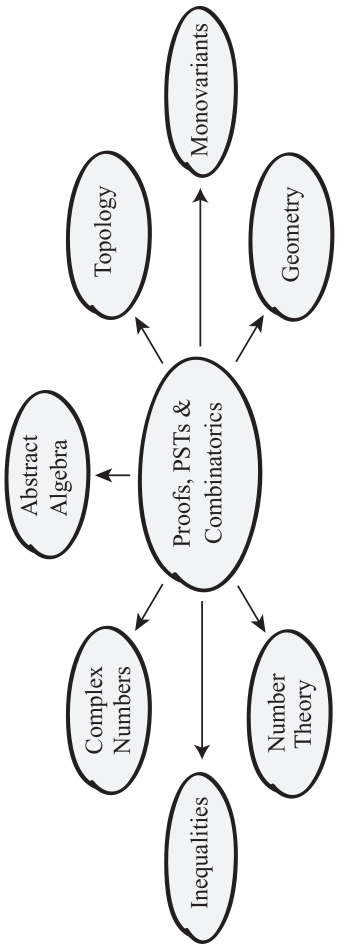


FIGURE 1. Main Areas in Volumes I-II

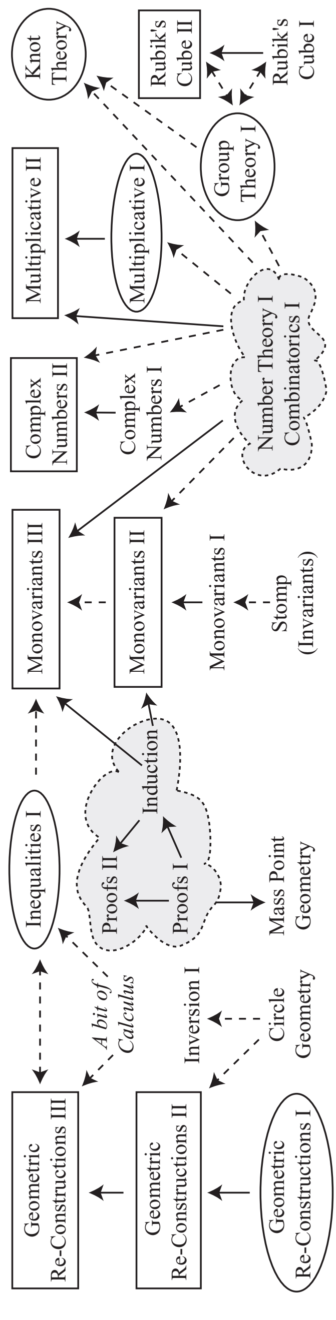


FIGURE 2. Logical Structure of the 24 Sessions in Volumes I-II

2.4. The middle or high school teacher who wishes to start a math circle in his/her school or teach a specially designed problem-solving class will find this book series invaluable. To start with, five sessions from volume I are a must for any math circle, as they provide techniques and a *foundation* for solid mathematical understanding; these are Combinatorics I, Number Theory I, Proofs I-II, and Induction.

Five of the topics in volume II are *introductory* and independent of each other; e.g., Geometric Re-Constructions I, Knot Theory, Group Theory I, Multiplicative I, and Inequalities I. Towards the end, some of these contain harder material suitable for *intermediate* level and the second-to-third year of a math circle. Four other sessions obviously need to be introduced after studying their *earlier counterparts*; e.g., Geometric Re-Constructions II, Rubik's Cube II, Monovariants II, and Complex Numbers II. The remaining three sessions are designed truly for the *advanced* reader: Multiplicative II, Monovariants III, and the last section of Geometric Re-Constructions III. *Open* questions or problems beyond the scope of the book are interspersed throughout the book and should be left to the *die-hards*.

Running a math circle, especially for a teacher, is a hard task. But it is possible. In the 1960's, Tom Rike (an editor for this book and a veteran high school math teacher) was working on his master's degree. While browsing in the library one day, he ran across *The USSR Olympiad Problem Book* [**74**]. It contained problems written for talented 7^{th}–10^{th} graders; yet, he could not solve any of these "elementary" problems. In his own words:

> *"My abstract algebra had been too* abstract*, and I did not have the concrete examples that I needed. I never took a class in number theory because it sounded too elementary. I had developed the real number system starting from the Peano axioms, but I didn't really understand the fundamentals of the natural numbers, prime numbers. This was an epiphany for me. I felt as though I had been challenged by some force outside me and did not know how to respond."*

For the next 30 years Tom studied olympiad problem solving, first on his own, then through workshops and math circles in the SF Bay Area. He ran his own math circle at Oakland High School and gave talks at just about all other circles around. Even though at times he was only "a few pages" ahead of the students, he kept on learning and teaching problem solving because working on math circles had come to be a large part of his life:

> *"Although I have not attained my goal of becoming a true olympiad problem solver, the journey I have made in pursuit of this goal has been one of the most rewarding endeavors in my life."*

Hence, a word to the middle and high school teachers: keep on reading the book, despite moments of difficulty or confusion. For the motivated, persevering, and caring teacher, there will come a time when he/she will look back at the material here, smile, and effortlessly deliver it to the students at his/her own math circle. Truly gratifying.

2.5. Proofs in particular. That proofs are important goes without question in the mind of Galileo's father:

> *"It appears to me that those who rely simply on the weight of authority to prove any assertion, without searching out the arguments to support it, act absurdly. I wish to question freely and to answer freely without any sort of adulation. That well becomes any who are sincere in the search for truth."*

In volume I we learned a variety of *proof methods*: by contradiction, Pigeonhole Principle, and induction; by counterexample, example, or general argument; using invariants or monovariants, and others. All sessions in volume II call for rigorous proofs. Although it is possible to get the gist of the sessions without being familiar with proofs, reviewing first the sessions on Proofs and Induction in volume I will make it faster and easier to read and understand this book.

2.6. The parent of a middle or high school student is also among our intended audience; in fact, parents are probably *the most important readers* because without their support and enthusiasm, without them bringing and encouraging their children, there would hardly be any top-tier math circles in the U.S. Hence, if you are among those parents or if you are a parent new to the math circle movement, this book series will provide a very strong beginning for your child. And for you as well.

As a parent, you can do three things with this book: give it to your child (but make sure that he/she has the necessary background – see the recommended basic books); learn from it and teach your child; or give it to his/her math teacher and encourage the founding of a school-based math circle.

Whatever path you choose to follow, it will eventually benefit your child and possibly a larger group of classmates. In any case, enjoy the book!

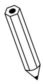
3. Notation and Technicalities

> *"Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures without which it is humanly impossible to understand a single word of it; without these one is wandering in a dark labyrinth."*
>
> Galileo

3.1. Marginalia. In addition to geometric "characters," we will also use a number of other symbols from algebra and logic. Let us examine first the non-standard margin icons which appear throughout the book.

 Warm-up or brute force

 Exercise

 Problem

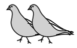
 Open question or one that requires extra knowledge

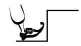
 Problem-solving technique

 Warning

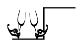
 Contradiction

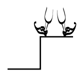
 Basic Pigeonhole Principle

 Generalized Pigeonhole Principle

 Basis step

 Inductive step

 Strong basis step

 Strong inductive step

The first four margin pictures refer to increasing difficulty of exercises and problems. Assigning such symbols is somewhat arbitrary since the same exercise could be *easy* for one person and could be a really *hard* problem for another; something may be beyond the knowledge of the reader *early* in the book, while *later* it may turn out to be a piece of cake. Thus, treat these symbols as a general guide to the difficulty of the material and make your own judgment after having attempted each problem.

The problem-solving techniques, indicated by an eye, are ubiquitous throughout the book and will be discussed in the next section. The warning road sign, the high-voltage symbol, and the pigeons were introduced in Proofs I in volume I. The last four margin pictures refer to the steps of basic and strong *mathematical induction*, the basis for Session 6 in volume I.

3.2. Logic. Mathematical statements that are proven are referred to by standard names such as *theorem, lemma, proposition, property,* or *corollary. Conjectures* are statements that are believed to be true, but no proof for them has been supplied yet. As opposed to volume I, in this book we will **not** avoid the formal *definition* environment; likewise, theorems and such will be often phrased formally.

All sessions have a section on *Hints and Solutions to Selected Problems.* There and throughout the text, you will see two symbols indicating the end of a solution. The standard square $\square$ indicates the end of a *complete* solution or a proof with minor gaps, which are usually mentioned and the reader is expected to easily fill them in. The diamond $\Diamond$ is at the end of an *incomplete* solution, partial proof, sketch of a proof, hint, or any discussion requiring more work by the reader to reach a complete proof.

The text uses standard mathematical words and expressions, such as "implies," "therefore," "if then," "only if," "if and only if," letter notations for various sets of numbers, e.g., $\mathbb{Z}$ for the integers, and many others. Even though some are explained and illustrated via examples, the reader is expected to be familiar with basic logic notions and notation (cf. the list of

Symbols and Notation on page 321). If you need to review or learn this material in depth, we refer you to the first chapter of Jacobs' *Geometry* [**43**] on *deductive reasoning.* A complete list of *Abbreviations* can be found on page 325.

3.3. Labeling and future volumes. Subfigures within the same figure are implicitly labeled in alphabetical order. For example, Figure 4 on page 9 contains subfigures Figure 4a, 4b, 4c, and 4d, reading from left to right. Finally, about half of the sessions are parts of series of sessions, to be continued in Volume III of the book.

4. The Art of Being a Mathematician and Problem Solving

> *"Perhaps I can best describe my experience of doing mathematics in terms of a journey through a dark unexplored mansion. You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture, but gradually you learn where each piece of furniture is. Finally, after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated. You can see exactly where you were."*
>
> Sir Andrew John Wiles

There are no manuals on how to become a mathematician. This book will give you tips and will point to possible paths; but the "art of being a mathematician" can be mastered only through personal experience. With every problem solved and every new definition or theorem learned, you will move closer to this goal. The two most important skills that you will acquire along the way are

- to think creatively while still "obeying the rules" and
- to make connections between problems, ideas, and even theories.

4.1. Problem-solving techniques. Although all sessions in this book are based on basic knowledge from middle and high school and are, therefore, accessible to a wide range of ages and mathematical backgrounds, to do the exercises, you need to develop *problem-solving techniques* (*PSTs*). Session 1 on inversion in volume I introduced PSTs as part of a trilogy of mathematical knowledge: Concepts, Theorems, and PSTs; and throughout this book you will encounter about 100 PSTs. You will also need to learn how to *fit together various mathematical parts* in order to move forward in the solutions.

4.2. Muddying your hands. Do not expect each session to be a collection of clearly spelled out recipes leading to instantaneous solutions .... Nope! The book will encourage you to apply the newly acquired knowledge to problems and will guide you along the way but will rarely give you ready answers. *"The best way to learn is to learn from your own mistakes,"* said my advisor Joe Harris. A number of places in the book will present common problem-solving pitfalls, and alternative ways to solve the same problem.

And so, it will be you, the reader, who has to commit to mastering the new math theories and techniques by

- "muddying your hands" in the problems,
- going back and reviewing necessary PSTs and theory, and
- persistently moving forward in the book.

Nothing good comes "for free": you will have to work hard, always with a pencil and paper in hand. Keep in mind that the math world is huge: you'll never know everything, but you'll learn *where* to find things, how to connect and use them. The rewards will be substantial.

5. Acknowledgments

5.1. Institutional support and sponsors. The Berkeley Math Circle was made possible through the years with the unwavering support of:

• *University of California at Berkeley Math Department*, which hosts the Circle and its web site and has provided student assistants and secretarial support every year since 1999. Through faculty grants, Ivan Matić has been able to act as an associate director. The department chairs Cal Moore, Hugh Woodin, Ted Slaman, Alan Weinstein, and Arthur Ogus have always been encouraging and supportive, and several dozen UCB professors have delivered Circle sessions.

• *Mathematical Sciences Research Institute*, which from its inception has overseen the project, provided funds through various sponsors, and hosted Circle meetings and events. Special thanks to Deputy Directors Hugo Rossi, Joe Buhler, Michael Singer, Bob Megginson, and Hélène Barchelo, Directors David Eisenbud and Robert Bryant, and Associate Director David Auckly for their leadership, understanding, and help.

A number of *sponsors* have financially supported BMC over the years: Packard Foundation, Toyota Foundation, Clay Mathematics Institute, Mosse Foundation for Art and Education, Merriam-Webster Foundation for the Scripps National Spelling Bee; National Science Foundation and other grants from Professors Ravi Vakil (Stanford), Bjorn Poonen, Alexander Givental, and Martin Olsson (UC Berkeley), and generous private donors.

5.2. Parents and students. *BMC parents* have encouraged and driven their kids to the Circle for years, brought snacks during the breaks, organized Circle parties, attended meetings, and donated time, effort, and personal funds to the Circle. We are especially grateful to Marc Whitlow, Mike Breen, Jennifer O'Dorney, Yuki Ishikawa, Ian Brown, and Tony DeRose for their enthusiasm, leadership, and professional services provided so selflessly to the Circle.

A sequence of UC Berkeley *student assistants* have contributed to the smooth operation of the Circle by communicating with circlers, parents, instructors, and administrators and by re-designing and maintaining the web site. Joyce Yeung, Maksim Maydanskiy, Wycee de Vera, William Chen, David Wertheimer, Michael Pejic, Stephanie Tung, and Hojae Lee, have been exceptionally professional and caring. Many thanks go to our *monthly contest coordinators*: Professors Alexander Givental and Bjorn Poonen, circlers Gabriel Carroll, Andrew Dudzik, Inna Zakharevich, Neil Herriot, Maksim Maydanskiy, Evan O'Dorney, Evan Chen, and former associate director Ivan Matić.

5.3. Professional support with the web site has been rendered on numerous occasions by Paulo de Souza, Dmitri Mironov, Steve Sizemore, and Igor Savine. Marsha Snow, Barbara Peavy, and Tom Brown have offered valuable *secretarial support* over the years. BMC owes its *logo design* to Archer Design, Inc.

As one can see, many dozens of people have been involved in running the Berkeley Math Circle: it is a joint operation born of the love and care for our young generation of mathematicians. The most important people in this operation are undoubtedly the **BMC instructors** (over 100), who have delivered the 800 sessions during the last 16 years. We would like to thank all of them! Twelve instructors joined BMC in the beginning and most have stayed with us throughout the years: Ted Alper, Tom Davis, Dmitry Fuchs, Alexander Givental, Quan Lam, Bjorn Poonen, Tom Rike, Vera Serganova, Tatiana Shubin, Zvezdelina Stankova, Paul Zeitz, and Joshua Zucker.

5.4. Book support. Edward Dunne, our AMS editor, and his staff have been very helpful in resolving technical and other issues. Gabriel Carroll is responsible for drawing some of the cartoons in the book series, inspired by the earlier BMC sessions. All USAMO problems are used with permission from the American Mathematics Competitions (AMC), Lincoln, Nebraska [**2**]. A few pictures and references have been taken from Wikipedia at `wikipedia.org/`.

With gratitude,
Zvezdelina Stankova
Berkeley Math Circle Director