Lecture 1

The Statistics of Topology and Algebra

> "I never heard of such a mathematician: he is actually a physicist."
>
> *Landau, on Poincaré*

> "It is not Shakespeare that most matters, but commentaries on his work."
>
> *A. P. Chekhov, as described by B. L. Pasternak*

Poincaré, the greatest mathematician of the recent era, divided all problems into two classes: binary problems and interesting problems. Binary problems are problems which admit of an answer "yes" or "no" (for example, Fermat's question).

Interesting problems are those for which an answer of "yes" or "no" is insufficient. They require investigation of questions that lead one further. For example, Poincaré was interested in how to change the conditions of a problem (for instance, the boundary conditions of a differential equation), while retaining the existence and uniqueness of its solution, or how the number of solutions varies when we make some other change. Thus he started the theory of bifurcations.

Three years before Hilbert gave his list of problems, Poincaré formulated the basic, in his view, mathematical questions that the nineteenth century would leave for the twentieth. This was the formulation of the mathematical basis for quantum and relativistic physics.

Today, many people think that relativistic physics at the time, in 1897, did not yet exist, since Einstein published his theory of relativity only in 1905. But Poincaré formulated the principle of relativity earlier, in his article of 1895, "On the Measurement of Time", which Einstein actually used (and which, by the way, he didn't acknowledge in writing until 1945). In just the same way, Schrödinger, in laying the foundation for quantum mechanics, achieved his success only because he used the mathematical works of his predecessor Hermann Weyl, whom no one mentioned later on, although Schrödinger actually references these works (in his first book).

1. Hilbert's Sixteenth Problem

Although I basically agree with Poincaré, today I will talk about a binary problem (or almost binary: this is why I am going to talk about of it) posed by Hilbert, the 16th in his list.

This problem is actually much older than Hilbert's list. In general, it is one of the fundamental problems of all of mathematical science (and of many of its applications).

Here is a very simple example: for an algebraic polynomial f in two variables x and y, we look at the curve along which it equals zero:

$$\{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$$

The problem consists in determining *the possible topological structures of this curve, if f is a polynomial of a given degree n.*

For example, if $n = 2$, then by the ancient theory of conic sections, the curve can be an ellipse, a hyperbola, a parabola, a pair of lines (which might possibly coincide), or the entire plane (if the polynomial is identically 0).

Augmenting the plane with points at infinity turns it into the projective plane, for which the problem becomes easier. (An ellipse, a hyperbola, and a parabola have the same structure in the projective plane. The only difference is in the position of this "circle" with respect to the line at infinity. See Figure 1.)

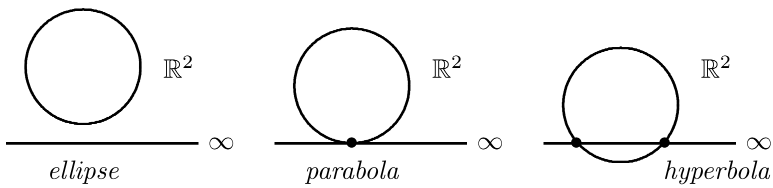

Figure 1

For $n > 2$ the question is more difficult, but Descartes and Newton had already analyzed the cases $n = 3$ and $n = 4$. Hilbert asserted the he had looked into it for curves of degree $n = 6$, but his result (the proof of which he never published) was erroneous.

According to a theorem of Harnack, a curve of degree n consists of no more than $g + 1 = \dfrac{(n-1)(n-2)}{2} + 1$ connected components (where g is the genus of the associated Riemann surface formed by the complex solutions of the equation of the curve in the complex projective plane $\mathbb{C}P^2$). According to a theorem in topology, every closed connected orientable surface is a surface of genus g, where g is the number of handles we must affix to a sphere in order to obtain this surface (see Figure 2).

For $n = 6$ we find that the genus of the Riemann surface g is 10, so that a real curve of degree 6 has no more than 11 components (which are called "ovals", and resemble circles, or at least are diffeomorphic to the circle S^1).

Hilbert asserted that if the number of ovals is maximal (that is, if there are 11 ovals), then these 11 ovals can be placed on the (projective) plane $\mathbb{R}P^2$ *in only two ways.*

Each oval bounds a "disk", diffeomorphic to the interior of a circle. (The complement of this disk in $\mathbb{R}P^2$ forms a Möbius band: this is how Möbius discovered his surface.)

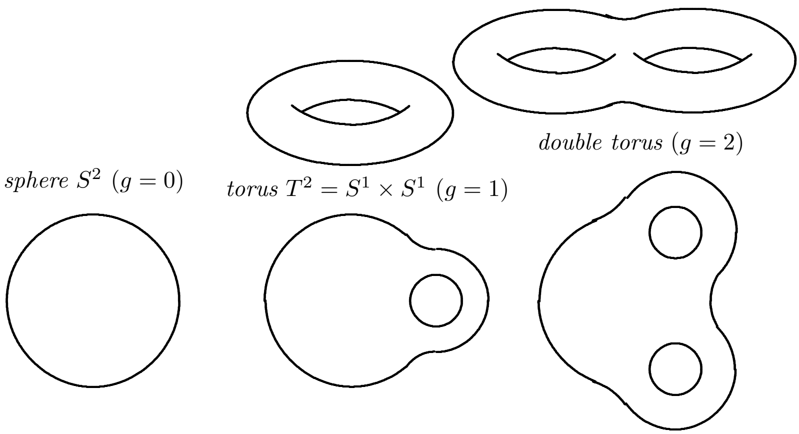

Figure 2. Surfaces of genus 0, genus 1, and genus 2.

And so, Hilbert asserted that only one of these disks can contain any other ovals inside it, and the number of interior ovals can only take on two values: 1 and 9 (Figure 3).

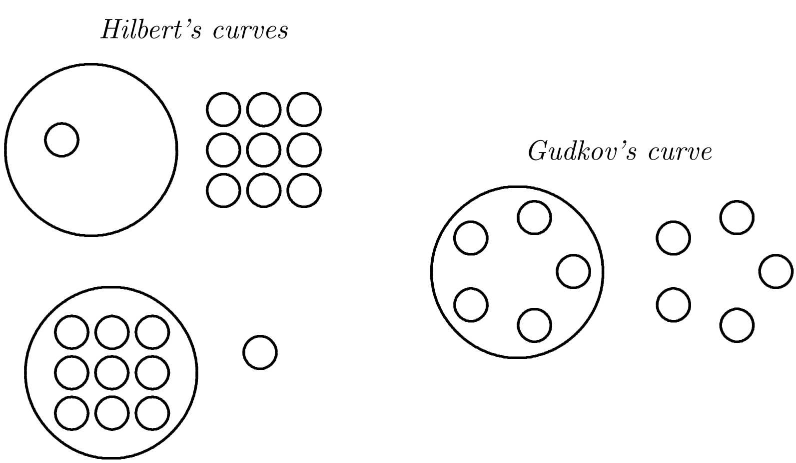

Figure 3. Algebraic curve of degree 6 with 11 ovals.

Hilbert's error consisted in the fact that the number of interior ovals could also be equal to 5. (This was discovered by Dmitri Andreevich Gudkov, a mathematician from Nizhny Novgorod, around 1970.) (See Editor's note 1, page 55.)

For curves of degree 8 Hilbert's question remains unanswered to this day: the 22 ovals of a curve of degree 8 can be placed on the plane in billions of different ways. But now certain bounds have been found which reduce the number of topologically distinct curves. There are now fewer than 90 cases. However, the number of examples actually constructed, while greater than 70, is not as large as the number of theoretical possibilities.

It is interesting that although the question seems to concern computational mathematics, our computers, so far, have contributed almost nothing to its solution.

If the coefficients of a polynomial are known, then it is possible for a computer to draw the positions of the ovals corresponding to the curve. But a count of *all* possibilities (for any values of the coefficients) is a much more difficult problem.

The problem also has an algorithmic solution, in the sense of mathematical logic. In principle, we can even find the number of connected regions into which the space of polynomials of degree n is divided by a bifurcation diagram, near which the type of the curve changes. But the number of computations needed for this is so large that no progress in computer technology will allow us the hope of a computer solution for the problem of polynomials of degree 8 in the foreseeable future.

Drifting a bit from the theme of today's lecture, I shall talk about one very recent success of computer technology that I know of, with regard to a closely related problem.

Let us think of the graph of a real polynomial of degree n in two variables as a surface, $z = f(x, y)$ in three dimensional space $\mathbb{R}^3$. Near some of its points the surface is locally convex. We call such points *elliptical* points. Around other points, the surface is locally saddle-shaped. We call these points *hyperbolic* points (see Figure 4).

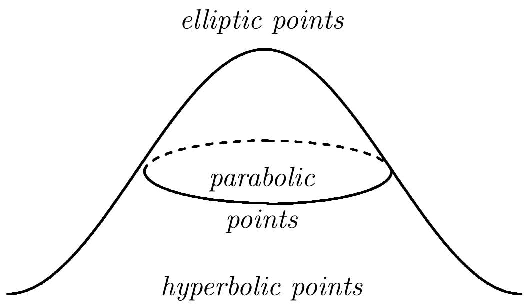

Figure 4. The parabolic curve on a smooth surface.

The elliptical and hyperbolic points are divided by a curve consisting of *parabolic points*. In terms of the partial derivatives of the function f, the curve of parabolic points is given by the equation

$$\det \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{vmatrix} = 0,$$

that is, by the condition $f_{xx} f_{yy} = (f_{xy})^2$, or that the Hessian of the function f is zero.

Let f be a polynomial of degree n. We may ask: *how many closed curves (ovals) can its parabolic curve be made up of?*

For a polynomial f of degree 4, the Hessian is also of degree 4, so by Harnack's theorem the number of ovals cannot exceed $g + 1 = 4$.

It is not hard to construct a polynomial of degree 4, with a parabolic curve consisting of three ovals. I leave this problem as an exercise for the reader.

But the problem of whether the parabolic curve of a polynomial of degree 4 can consist of four ovals turns out to be very difficult.

It was solved in Mexico in 2005 by Adriana Ortiz-Rodriguez, who defended her dissertation in Paris as my student. In her dissertation, she proved that the number of ovals in the parabolic curve of a polynomial of degree n is bounded from above by an^2 and below by bn^2, where $a > b$.

When she was still a student (in the University of Paris, Jussieu), she came into my seminar and asked for a problem. I said that to understand my problems, one must first solve the 100 problems of my "Mathematical Trivium" [**1**]. Good students in Moscow solve them all in 3 hours.

Adriana brought me solutions to these problems, but they all turned out to be wrong. She asked for a week's time to think about them, after which she brought in 10 correctly solved problems. After 10 weeks, she solved all 100 of them and started making sense of mathematics.

But when I wanted to formulate a research problem for her, Adriana said, "No, I thought up a problem of my own, in the style of your seminar and of your students' work on Lagrangian singularities in symplectic geometry." And she formulated the problem which appears above, about parabolic curves.

I replied that I was now convinced that in Mexico they study mathematics as badly as in Paris (where I knew very well how low the level of the students was).

Adriana's inability to solve the problems of the Trivium was the legacy of just this bad training in the basics of mathematics that she had been subjected to, both in Mexico and in Paris. With regard to her imagination and mathematical capability, everything was fine (as her further experience with the Trivium, and with parabolic curves showed). After I taught her everything using my 100 problems, she became an excellent mathematician.

The question of the growth of the number of ovals of a parabolic curve for polynomials of degree n (the question of how the constants a and b in the asymptotic values an^2 and bn^2 from above and below approach each other) remains open to this day. And this is why I included it in this lecture, hoping that I might find some talented students here as well.

As for the original value $n = 4$, Adriana defended her dissertation in Paris, then became a professor in Mexico, where she had unlimited access to computers. In a year of uninterrupted work, the central processing unit of her computer examined 50 million polynomials $f(x, y)$

of degree 4. For three of them, the parabolic curve turned out to consist of four ovals.

When the coefficients of a polynomial are known, the number of ovals in its parabolic curve can be found in a matter of minutes, even without a computer. So the proof of the theorem does not require any mention of a computer experiment. But *finding* these remarkable polynomials without a computer is hopeless, so the application of the computer experiment to the difficult solution of the problem turns out to be decisive.

I hope that my readers will be able to achieve analogous successes in solving the problems we discuss below.

Remark. Before going further, I will explain several things which have been used above, and which are carefully hidden from the students in the traditional pseudo-scientific exposition of mathematics.

The *determinant* of a second-order matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the *area of the parallelogram* constructed using the column vectors (a, c) and (b, d) (see Figure 5), which is written with a plus sign if the vectors orient the plane in the same way as the first and second unit vectors along the axes (and with a minus sign otherwise).

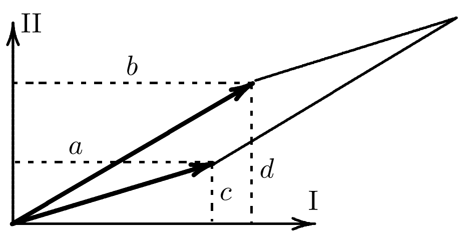

Figure 5. A positively oriented parallelogram.

Two pairs of linearly independent vectors orient the plane *in the same way* if we can connect them with a continuous path in space of ordered pairs of linearly independent vectors on the plane.

There are *exactly two different orientations* (that is, classes of equivalent ordered pairs of vectors on the plane, or ordered frames of n linearly independent vectors, in $\mathbb{R}^n$, for any n). This most important natural scientific fact (which is the only explanation for the strange rule "minus times minus gives plus") is usually kept secret from students, and all this geometry is replaced with the postulate that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

which is, in fact, an easy *consequence* of the topological fact given above. (It is useful also to mention the *linear* dependence of the determinant on each column vector, and its skew-symmetry: the fact that its sign changes when we interchange two columns.)

The second derivatives of a polynomial (or any other smooth function) at a given point form a matrix (of order m for a function of m variables), $\partial^2 f/\partial x_i \partial x_j$. The determinant of this *Hessian* matrix for the function f is simply called the *Hessian* of the function f. It is useful to note that the sign of the Hessian of a function f coincides with the sign of the Gaussian curvature of the graph of the function f (and, by the way, does not depend on the choice of orientation of the space where the function is defined). I will not dwell on this remark because it is meaningful only to those familiar with the sign of the Gaussian curvature, and those to whom it is meaningful can easily prove the assertions made above about the relationship of the Hessian to the Gaussian curvature of a graph.

Let me make one more note, about the genus g of the Riemann surface of an algebraic curve of degree n. We have used, above, the "Riemann-Hurwitz formula":

$$g = \frac{(n-1)(n-2)}{2}.$$

For instance, curves of degree $n = 1$ (a line) and $n = 2$ (a circle) have genus 0; that is, there is a real diffeomorphism between them and the sphere S^2 (also called the Riemann sphere $\mathbb{C} \sqcup \{\infty\}$ or the complex projective line $\mathbb{C}P^1$).

For a line this is clear, and for a circle this follows from the rational parametrization using the "tangent of half an angle" formula

$t = \tan\beta = y/(1+x)$:

$$(1) \qquad x = \frac{1-t^2}{1+t^2}, \; y = \frac{2t}{1+t^2} \quad \text{for } x^2+y^2=1.$$

It is a useful problem to try to understand the topological structure of the "complex sphere" given in projective space $\mathbb{C}P^3$ by the affine equation $x^2+y^2+z^2=1$. Answer: it is a four-dimensional manifold diffeomorphic to the direct product of two ordinary spheres, $S^2 \times S^2$.

The formulas given in (1) define a diffeomorphism of the complex circle to the sphere S^2. (They also give us the "Egyptian right triangles", $3^2+4^2=5^2$, $5^2+12^2=13^2$ and so on, namely, $a^2+b^2=c^2$ for $x=a/c, y=b/c$, where, according to the formula (1) with $t=u/v$, $a=v^2-u^2, b=2uv, c=u^2+v^2$.)

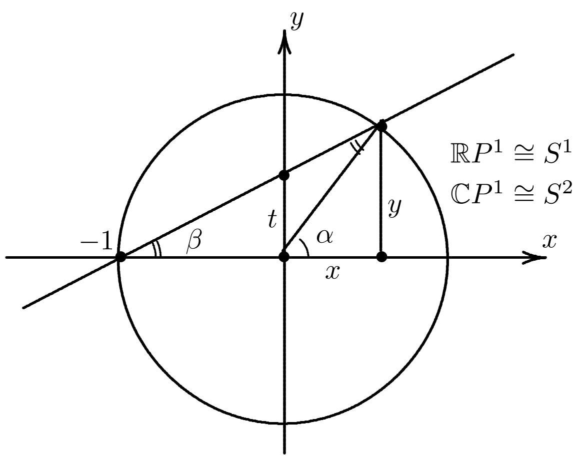

Figure 6. The rational parametrization of a circle.

We can derive formula (1) as follows (Figure 6). We draw the line $\{y=t(x+1)\}$ through the point $(x=-1,\ y=0)$ in the plane. Substituting this value of y into the equation $x^2+y^2=1$ of the circle, we find the corresponding abscissa x of the point of intersection of the line with the circle by solving a certain quadratic equation, one root of which $(x=-1)$ we already know.

For the other root, Vieta's Theorem (the formula for the sum of the roots of a quadratic equation) gives us a rational expression in t, from which we can derive the first (and then the second) formula of the parametrization (1).

Instead of all this algebra, we can use the geometric identity $\alpha = 2\beta$ from the theorem about the exterior angle (of an isosceles triangle):

$$x = \cos\alpha, \ \ y = \sin\alpha, \ \ t = \tan\beta.$$

For those who know some calculus, I note also that from the same rational parametrization of the circle we can get an easy computation (in elementary functions) of any Abelian integral along a circle:

$$I = \int\limits_{x^2+y^2=1} R(x,y)dx,$$

where R is a rational function.

Indeed, the rational parametrization (1) reduces the computation of the integral I to the integration of a rational function of the parameter t,

$$I = \int r(t)dt.$$

Abel proved that *it is impossible to perform this integration in elementary functions if the Abelian integral is taken not along the circle but along some curve of higher genus (with $g > 1$).* For example, this is impossible even for elliptic integrals (along a curve of the form $y^2/2 + U(x) = 0$, where U is a polynomial of degree 3, for example $U(x) = x^3 + ax + b$).

The proof of this topological theorem of Abel is also a marvellous exercise.

It is *topological* because it is not just the function[1]

$$t(X) = \int^X \frac{dx}{y}, \text{ where } \frac{y^2}{2} + U(x) = 0$$

that cannot be presented as a finite combination of elementary functions. In fact such a representation is impossible for any function

[1] I.e., a function which expresses the time t it takes to move to a position X under the actions described by Newton's equation: $\dfrac{d^2x}{dt^2} = -\dfrac{dU}{dx}$.

which is topologically equivalent to the (multi-valued) complex function t. And such a representation is also impossible for the inverse functions, which are equivalent to the "elliptic function" $X(t)$ (for non-degenerate value of the coefficients a and b).

The Riemann-Hurwitz formula ($g = (n-1)(n-2)/2$ for smooth curves of degree n) is easiest to prove using the following "Italianate" argument.

Consider any natural family of algebraic-geometric complex objects. An example might be the family of polynomials of degree n in one variable. Other examples are the family of polynomials of a given degree in two variables, which give us algebraic curves, or the corresponding family of homogeneous polynomials of fixed degree in three variables, which give us algebraic curves in the complex projective plane $\mathbb{C}P^2$.

The "Italianate" reasoning consists in noting that *all the non-degenerate objects in this family are topologically identical.* For example, all polynomials of degree n in one variable without multiple roots have the same number of roots; all smooth algebraic curves of degree n in $\mathbb{C}P^2$ have the same genus $g(n)$, independent of the particular choice of a curve.

The proof of this observation is topological. The idea is that the degeneracy of a complex object is determined by a complex equation (the discriminant is equal to zero, in the case of a polynomial in one variable, and so on). And this complex condition on complex coefficients, whose choice gives us an object of the family, gives rise to *two* independent equations in real numbers (both the real and the imaginary parts of the discriminant must vanish).

Therefore *the algebraic variety of all the degenerate objects has a real codimension of two* (in the case of the family of polynomials we're looking at, and so on). But a subvariety with real codimension two cannot divide a smooth variety of all objects of the family into parts (just as a point cannot divide a plane, and a line or curve cannot divide three-dimensional space into parts).

Therefore the variety of non-degenerate objects is connected. From this it follows that all these non-degenerate objects have the same topological type, since if we move along a curve in the space of non-degenerate objects (for example, polynomials without multiple roots), the topological structure of the object (the number of roots of the equation, in the example just given) will not change (by the implicit function theorem).

The principle just demonstrated shows that *to compute the topological characteristics of all non-degenerate objects in a complex family it is enough to look at one example and compute these characteristics for this one object: for all other non-degenerate objects the characteristics will be the same.*

For example, in the case of polynomials of degree n it is enough to take the polynomial

$$f(x) = (x-1)(x-2)\cdots(x-n),$$

which obviously has the n roots $x = 1, 2, \ldots, n$ (each with multiplicity 1).

By the "Italianate Principle", a consequence of this observation is the "Fundamental Theorem of Algebra": *every polynomial of degree n in one variable without multiple roots has exactly n complex roots.*

In the case of planar algebraic curves it is sufficient to find the genus of one (non-singular) curve of degree n.

Let us begin the topological investigation with a singular curve of degree n, which breaks into n lines, intersecting in pairs at $n(n-1)/2$ distinct points (Figure 7).

If the equation of this curve has the form $f_0 = 0$, where f_0 is the product of n linear non-homogeneous functions $a_j x + b_j y + c_j$ then the equation $f_\varepsilon = 0$ (where $f_\varepsilon = f_0 - \varepsilon$) gives us a smooth curve of degree n for small values of $\varepsilon \neq 0$. We now compute the genus of this curve.

This computation proceeds as follows. The curve $f_0 = 0$ is formed by n spheres S_j^2, which intersect in pairs at $n(n-1)/2$ distinct points. For small ε, the transition to the curve $f_\varepsilon = 0$ requires changing the cross formed by two transversally intersecting smooth spheres near their point of intersection into a cylinder connecting the completions of neighborhoods of the points of intersection on each sphere (Figure 8).

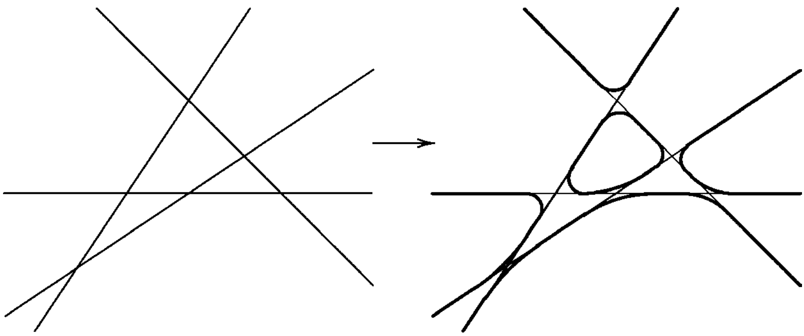

Figure 7. A deformation of a decomposable curve.

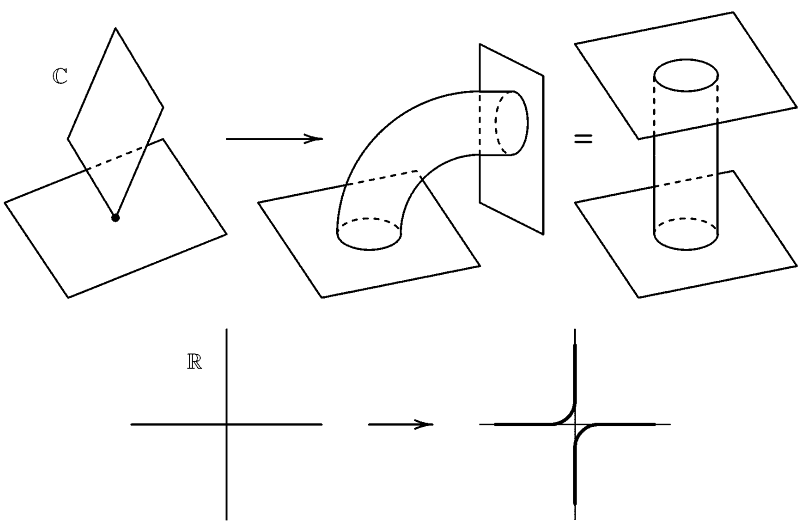

Figure 8. The deformation at a double point.

Let us find out how many handles are created after $n(n-1)/2$ such independently completed deformations (near each point of intersection).

Let us take one of the n spheres $S_1^2, \ldots, S_n^2$, say S_1^2. After our deformation, its $n-1$ points of intersection with the other spheres connect each of the remaining spheres with the first, so that together they form another surface Σ^2 which is again diffeomorphic to the sphere, except that we haven't taken into account the remaining $(n-2)+(n-3)+\cdots+1=(n-1)(n-2)/2$ points of intersection of the other spheres with each other (Figure 9).

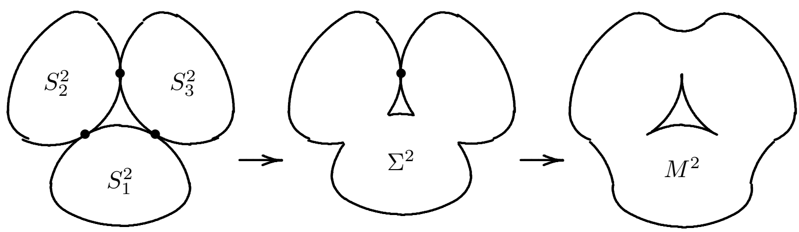

Figure 9. Constructing a surface from a set of spheres.

After transforming the $(n-1)(n-2)/2$ points of self-intersection of the surface Σ^2 using the same number of tubes, we turn the "sphere" Σ^2 into a (smooth) surface M^2 which is a sphere with handles. The number of handles g is the number of tubes which replaced the points of self-intersection of the "sphere" Σ^2, so $g = (n-1)(n-2)/2$.

Figure 9 shows the case $n = 3$, where we get $g = 1$. Hence the surface M^2 turns out to be a torus, which is of genus 1.

Thus we obtain the Riemann-Hurwitz formula $g = \dfrac{(n-1)(n-2)}{2}$.

Harnack's inequality, the assertion that a real curve of genus g has no more than $g+1$ ovals, turns out to be a special case of *Smith's inequality*:

$$\sum b_k(M_{\mathbb{R}}) \leq \sum b_k(M_{\mathbb{C}}). \tag{2}$$

Here, $M_{\mathbb{C}}$ is a complex algebraic variety (for example, the Riemann surface of a curve). If this variety is given by an equation with real coefficients, then it is acted on by the symmetry ("involution") $\sigma : M_{\mathbb{C}} \longrightarrow M_{\mathbb{C}}$ of complex conjugation (the replacement of a point with complex coordinates $z_j = x_j + iy_j$ by the point with complex coordinates $\overline{z_j} = x_j - iy_j$). Obviously, $\sigma^2 = 1$, and the real variety $M_{\mathbb{R}}$ consists of the fixed points of the involution σ (which are ovals, in the case of a curve).

The numbers b_k in inequality (2) are "the Betti numbers" for the chains with coefficients in the group $\mathbb{Z}_2$ of two elements.

The Betti numbers for the circle are

$$b_0 = b_1 = 1, \quad b_k = 0 \text{ for } k > 1.$$

For a Riemann surface of genus g we have

$$b_0 = b_2 = 1, \quad b_1 = 2g, \quad b_k = 0 \text{ for } k > 2.$$

Here the $2g$ one-dimensional cycles are the "parallels" and "meridians" of the g handles.

Thus, in the case of curves of genus g, Smith's inequality takes the form

$$2 \text{ (number of ovals)} \leq 2g + 2;$$

that is, we have obtained Harnack's inequality:

$$\text{number of ovals} \leq g + 1.$$

The proof of Smith's inequality itself is not that difficult if we look at the action of the involution σ on all possible chains (of a triangulation of the manifold which is symmetric with respect to the involution σ). In the case of Riemann surfaces of real curves the most important argument of Smith's theory lies in the fact that among its ovals there can exist *only one* homological relationship (the sum of the ovals is homological to zero) in the one-dimensional homology of the Riemann surface: otherwise this surface would not be connected, but would fall into connected two-dimensional components (made up of pieces of the form a and σa, where the boundary of the two-dimensional chain a is the left part of the relation between the ovals).

To these remarks from real algebraic geometry I can add that in addition to graphs of polynomials, Adriana Ortiz-Rodriguez, in her dissertation, also examined parabolic curves on any algebraic surface of degree n in the real three-dimensional projective space $\mathbb{R}P^3$. In this case, the number of parabolic curves is bounded by her from above and below by the quantities an^3 and bn^3, where the constant a is approximately 10 times greater than the constant b.

I formulate this result because I have hopes that my audience might want to find the exact rate of growth of the number of parabolic curves, by moving a and b closer together.

The results of Gudkov about curves of degree 6 have laid the foundation for a remarkable new theory connecting the real algebraic geometry of Hilbert's 16th problem with quantum field theory and with multi-dimensional topology.

The fact that the number of interior ovals in Gudkov's list of curves of degree 6 with 11 ovals (1, 5 and 9) increases by 4 is not accidental. Passing from the curve $f(x, y) = 0$ to the bounded surface with boundary M: $f(x, y) \geq 0$, we come to a sequence of Euler characteristics which differ by 8.

Analyzing the real projective algebraic curves of degree $n = 2k$ found by Gudkov which have the largest possible number of ovals according to Harnack's theorem, I noticed that for these curves, the Euler characteristics of the surfaces M satisfy the congruence

$$\chi(M) \equiv k^2 \pmod{8}, \tag{3}$$

which I call "Gudkov's congruence". (See Editor's note 2, page 56.)

Congruences modulo 8 (for the signatures of intersection forms) turn out to be standard in the topology of four-dimensional closed varieties, so I started searching for four-dimensional manifolds in the topology of (*one-dimensional*) real algebraic curves.

These manifolds turned out to be *complexifications of surfaces with boundaries* M^2. To complexify a surface given by an inequality $f(x, y) \geq 0$, I write down this geometric inequality in the algebraic form $f(x, y) = z^2$. In the complex domain, this formula defines a manifold of real dimension four: a two-sheet covering of the complement of the Riemann surface (of the complex curve) $f(x, y) = 0$ in $\mathbb{C}P^2$, branching along this Riemann surface.

Applying topological results about the divisibility by 8 of the signature to this four-dimensional variety, I proved congruence (3) modulo 4. Then Rokhlin, using deeper results from differential topology of smooth four-dimensional varieties, proved Gudkov's congruence (3) itself.

It is interesting that Gudkov himself, whom I informed of this congruence while writing a report on his dissertation, thought that it was false, since he believed he knew counter-examples to it (which, however, turned out to be as false as Hilbert's results about curves of degree 6, which had been disproved in this very dissertation.)

Recently, congruence (3) has become the foundation for a large number of new results in real algebraic geometry, in differential topology, and even in quantum field theory. But, unfortunately these results are not sufficient even for the classification of topological structures of curves of degree 8 in Hilbert's 16th problem.

Returning to the 16th problem, I note that Hilbert, in my view, left out the most important questions in formulating it.

The point is that topological structures can be different not only for real algebraic curves (of a given degree) $\{f(x, y) = 0\}$, but also for polynomials $f\colon \mathbb{R}^2 \to \mathbb{R}$ defining these curves.

In formulating his problem, Hilbert should have included not just the question of topological classification of curves of a given degree in the real projective plane, but also the question of the *topological classification of the polynomials defining these curves.*

This problem has not been solved, as far as I know, even for curves of degree $n = 4$ (which Descartes had already classified). In the next section I will discuss the topological classification of functions and polynomials, where much remains unknown (which is partly Hilbert's fault).

2. The Statistics of Smooth Functions

To describe the topological structure of a smooth real function, we associate it with the graph whose points are the connected components of the level hypersurfaces of this function.

For a non-degenerate "Morse Function" $f\colon S^n \to \mathbb{R},\ n > 1$ (see Editor's note 3, page 56), this graph turns out to be a tree with T triple branch points, $K = T+2$ end vertices or 'leaves' and $P = 2T+1$ edges that connect $K + T = 2T + 2$ vertices of the graph.

Example 1. For the "Mount Elbrus Function" (which shows the altitude at each point on the mountain), there are two local maxima,