Part 4

Problems for Children 5 to 15 Years Old

Problems

I wrote these problems in Paris in the spring of 2004. Some Russian residents of Paris had asked me to help cultivate a culture of thought in their young children. This tradition in Russia far surpasses similar traditions in the West.

I am deeply convinced that this culture is developed best through early and independent reflection on simple, but not easy, questions, such as are given below. (I particularly recommend Problems 1, 3, and 13.)

My long experience has shown that C-level students, lagging in school, can solve these problems better than outstanding students, because the survival in their intellectual "Kamchatka" at the back of the classroom "demanded more abilities than are requisite to govern Empires", as Figaro said of himself in the Beaumarchais play. A-level students, on the other hand, cannot figure out "what to multiply by what" in these problems. I have even noticed that five year olds can solve problems like this better than can school-age children, who have been ruined by coaching, but who, in turn, find them easier than college students who are busy cramming at their universities. (And Nobel prize or Fields Medal winners are the worst at all in solving such problems.)

1. Masha was seven kopecks short of the price of an alphabet book, and Misha was one kopeck short. They combined their money to buy one book to share, but even then they did not have enough. How much did the book cost?

2. A bottle with a cork costs $1.10, while the bottle alone costs 10 cents more than the cork. How much does the cork cost?

3. A brick weighs one pound plus half a brick. How many pounds does the brick weigh?

4. A spoonful of wine from a barrel of wine is put into a glass of tea (which is not full). After that, an equal spoonful of the (non-homogeneous) mixture from the glass is put back into the barrel. Now there is a certain volume of "foreign" liquid in each vessel (wine in the glass and tea in the barrel). Is the volume of foreign liquid greater in the glass or in the barrel?

5. Two elderly women left at dawn, one traveling from A to B and the other from B to A. They were heading towards one another (along the same road). They met at noon, but did not stop, and each of them kept walking at the same speed as before. The first woman arrived at B at 4 PM, and the second arrived at A at 9 PM. At what time was dawn on that day?

6. The hypotenuse of a right-angled triangle (on an American standardized test) is 10 inches, and the altitude dropped to it is 6 inches. Find the area of the triangle.

American high school students had been successfully solving this problem for over a decade. But then some Russian students arrived from Moscow, and none of them was able to solve it as their American peers had (by giving 30 square inches as the answer). Why not?

7. Victor has 2 more sisters than he has brothers. How many more daughters than sons do Victor's parents have?

8. There is a round lake in South America. Every year, on June 1, a Victoria Regia flower appears at its center. (Its stem rises from the bottom, and its petals lie on the water like those of a water lily). Every day the area of the flower doubles, and on July 1, it finally covers the entire lake, drops its petals, and its seeds sink to the bottom. On what date is the area of the flower half that of the lake?

9. A peasant must take a wolf, a goat and a cabbage across a river in his boat. However the boat is so small that he is able to take only one of the three on board with him. How can he transport all three across the river? (The wolf cannot be left alone with the goat, and the goat cannot be left alone with the cabbage.)

10. During the daytime a snail climbs 3cm up a post. During the night it falls asleep and slips down 2cm. The post is 10m high, and a delicious sweet is waiting for the snail on its top. In how many days will the snail get the sweet?

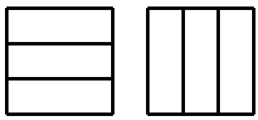
11. A hunter walked from his tent 10 km. south, then turned east, walked straight eastward 10 more km, shot a bear, turned north and after another 10 km found himself by his tent. What color was the bear and where did all this happen?

12. High tide occurred today at 12 noon. What time will it occur (at the same place) tomorrow?

13. Two volumes of Pushkin, the first and the second, are side-by-side on a bookshelf. The pages of each volume are 2cm thick, and the front and back covers are each 2mm thick. A bookworm has gnawed through (perpendicular to the pages) from the first page of volume 1 to the last page of volume 2. How long is the bookworm's track? [This topological problem with an incredible answer–4 mm–is totally impossible for academicians, but some preschoolers handle it with ease.]

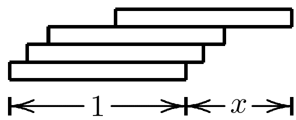
14. Viewed from above and from the front, a certain object (a polyhedron) gives the shapes shown. Draw its shape as viewed from the side. (Hidden edges of the polyhedron are to be shown as dotted lines.)

15. How many ways are there to break the number 64 up into the sum of ten natural numbers, none of which is greater than 12? Sums which differ only in the order of the addends are not counted as different.

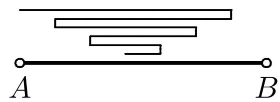


Top view Front view

To Problem 14



To Problem 16



To Problem 17

16. We have a number of identical bars (say, dominoes). We want to stack them so that the highest hangs out over the lowest by a length equal to x bar-lengths. What is the largest possible value of x?

17. The distance between towns A and B is 40km. Two cyclists leave from A and B simultaneously traveling towards one another, one at a speed of 10km/h and the other at a speed of 15km/h. A fly leaves A together with the first cyclist, and flies towards the second at a speed of 100km/h. The fly reaches the second cyclist, touches his forehead, then flies back to the first, touches his forehead, returns to the second, and so on until the cyclists collide with their foreheads and squash the fly. How many kilometers has the fly flown altogether?

18. Vanya solved a problem about two pre-school age children. He had to find their ages (which are integers), given the product of their ages.

Vanya said that this problem could not be solved. The teacher praised him for a correct answer, but added to the problem the condition that the name of the older child was Petya. Then Vanya could solve the problem right away. Now you solve it.

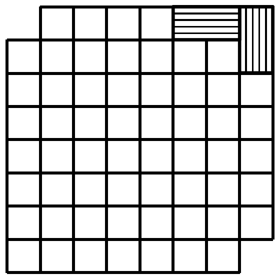
19. Is the number 140359156002848 divisible by 4206377084?

20. One domino covers two squares of a chessboard. Cover all the squares except for its two opposite corners (on the same diagonal) with 31 dominoes. (A chessboard consists of $8 \times 8 = 64$ squares.)

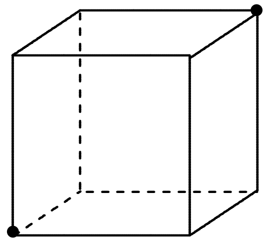
21. A caterpillar wants to slither from the front left corner of the floor of a cubical room to the opposite corner (the right rear corner of the ceiling). Find the shortest route for such a journey along the walls of the room.

22. You have two vessels of volumes 5 liters and 3 liters. Measure out one liter, leaving the liquid in one of the vessels.

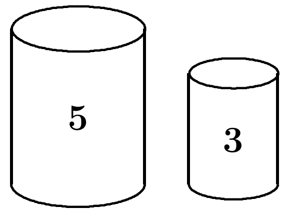
23. There are five heads and fourteen legs in a family. How many people and how many dogs are in the family?



To Problem 20



To Problem 21



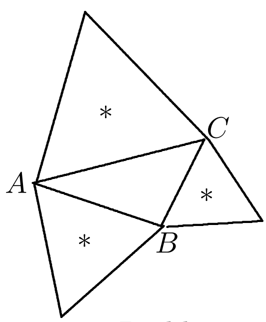
To Problem 22

24. Equilateral triangles are constructed externally on sides AB, BC, and CA of a triangle ABC. Prove that their centers (marked by asterisks on the diagram) form an equilateral triangle.

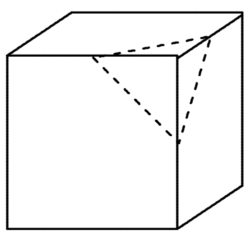
25. What polygons may be obtained as sections of a cube cut off by a plane? Can we get a pentagon? A heptagon? A regular hexagon?

26. Draw a straight line through the center of a cube so that the sum of the squares of the distances to it from the eight vertices of the cube is (a) maximal, (b) minimal (as compared with other such lines).

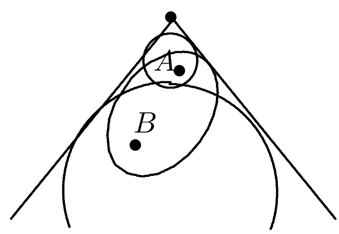
27. A right circular cone is cut by a plane along a closed curve. Two spheres inscribed in the cone are tangent to the plane, one at point A and the other at point B. Find a point C on the cross-section such that the sum of the distances $CA + CB$ is (a) maximal, (b) minimal.



To Problem 24



To Problem 25



To Problem 27

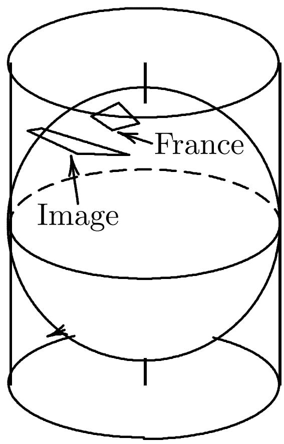
28. The Earth's surface is projected onto a cylinder formed by the lines tangent to the meridians at the points where they intersect the equator. The projection is made along rays parallel to the plane of the equator and passing through the axis of the earth that connects its north and south poles. Will the area of the projection of France be greater or less than the area of France itself?

29. Prove that the remainder upon division of the number 2^{p-1} by an odd prime p is 1 (for example: $2^2 = 3a + 1$, $2^4 = 5b + 1$, $2^6 = 7c + 1$, $2^{10} - 1 = 1023 = 11 \cdot 93$).

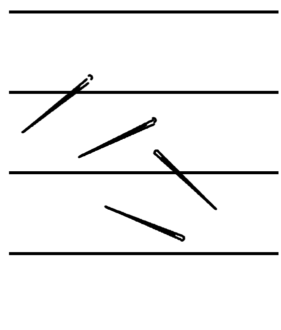
30. A needle 10 cm. long is thrown randomly onto ruled paper. The distance between neighboring lines on the paper is also 10 cm. This is repeated N (say, a million) times. How many times (approximately, up to a few per cent error) will the needle fall so that it intersects a line on the paper?

One can perform this experiment with $N = 100$ instead of a million throws. (I did this when I was 10 years old.)

[The answer to this problem is surprising: $\frac{2}{\pi}N$. Moreover, even for a curved needle of length $a \cdot 10$cm. the number of intersections observed over N throws will be approximately $\frac{2a}{\pi}N$. The number $\pi \approx \frac{355}{113} \approx \frac{22}{7}$.]



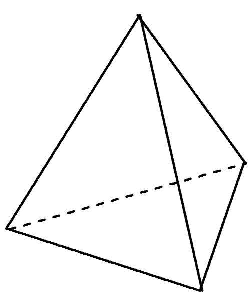
To Problem 28



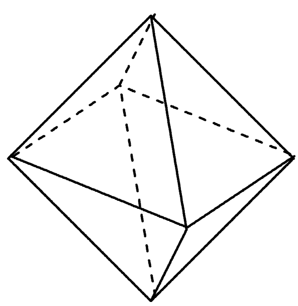
To Problem 30

31. Some polyhedra have only triangular faces. Some examples are the Platonic solids: the (regular) tetrahedron (4 faces), the octahedron (8 faces), and the icosahedron (20 faces). The faces of the icosahedron are all identical, it has 12 vertices, and it has 30 edges.

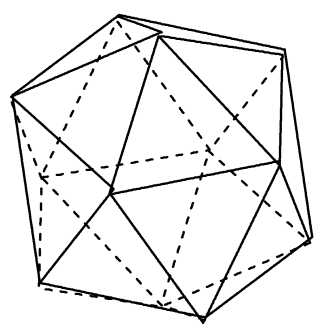
Is it true that for any such solid (a bounded convex polyhedron with triangular faces) the number of faces is equal to twice the number of vertices minus four?



tetrahedron (tetra = 4)



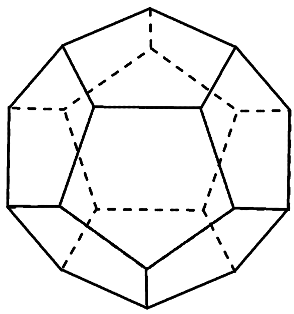
octahedron (octa = 8)



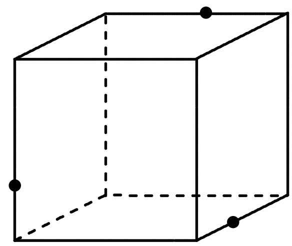
icosahedron (icosa = 20)

32. There is one more Platonic solid (there are 5 of them altogether): a dodecahedron. It is a convex polyhedron with twelve (regular) pentagonal faces, twenty vertices and thirty edges (its vertices are the centers of the faces of an icosahedron).

Inscribe five cubes in a dodecahedron, whose vertices are also vertices of the dodecahedron, and whose edges are diagonals of faces of the dodecahedron. (A cube has 12 edges, one for each face of the dodecahedron). [This construction was invented by Kepler to describe his model of the planets.]



To Problem 32



To Problem 33^{bis}

33. Two regular tetrahedra can be inscribed in a cube, so that their vertices are also vertices of the cube, and their edges are diagonals of the cube's faces. Describe the intersection of these tetrahedra.

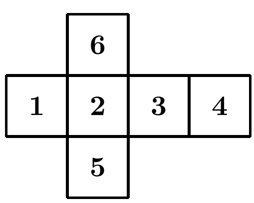
What fraction of the cube's volume is the volume of this intersection?

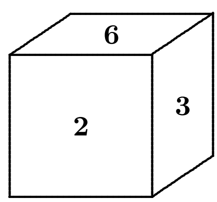
$\mathbf{33^{bis}}$. Construct the section cut of a cube cut off by the plane passing through three given points on its edges. [Draw the polygon along which the plane intersects the faces of the cube.]

34. How many symmetries does a tetrahedron have? A cube? An octahedron? An icosahedron? A dodecahedron? A symmetry of a figure is a transformation of this figure preserving lengths.

How many of these symmetries are rotations, and how many are reflections in planes (in each of the five cases listed)?

35. How many ways are there to paint the six faces of similar cubes with six colors (1,. . . ,6) [one color per face] so that no two of the colored cubes obtained are the same (that is, no two can be transformed into each other by a rotation)?

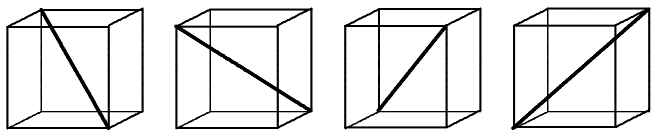




36. How many different ways are there to permute n objects?

For $n = 3$ there are six ways: $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(3, 2, 1)$. What if the number of objects is $n = 4$? $n = 5$? $n = 6$? $n = 10$?

37. A cube has 4 major diagonals (that connect its opposite vertices). How many different permutations of these four objects are obtained by rotations of a cube?



38. The sum of the cubes of several integers is subtracted from the cube of the sum of these numbers. Is this difference always divisible by 3?

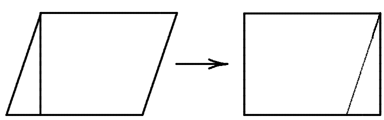
39. Answer the same question for the fifth powers and divisibility by 5, and for the seventh powers and divisibility by 7.

40. Calculate the sum

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{99 \cdot 100}$$

(with an error of not more than 1% of the correct answer).

41. If two polygons have equal areas, then they can be cut into a finite number of polygonal parts which may then be rearranged to obtain both the first and second polygons. Prove this. [For spatial solids this is not the case: the cube and the tetrahedron of equal volumes cannot be cut this way!]



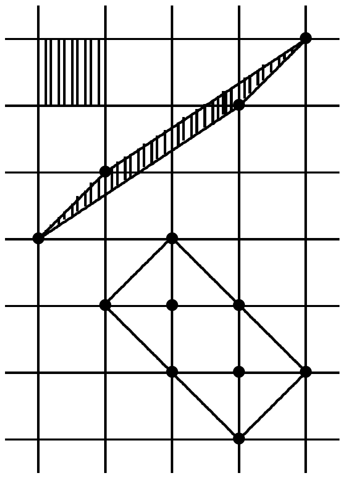
42. Four lattice points on a piece of graph paper are the vertices of a parallelogram. It turns out that there are no other lattice points either on the sides of the parallelogram or inside it. Prove that the area of such a parallelogram is equal to that of one of the squares of the graph paper.

43. Suppose, in Problem 42, there turn out to be a lattice points inside the parallelogram, and b lattice points on its sides. Find its area.

44. Is the statement analogous to the result of problem 43 true for parallelepipeds in 3-space?

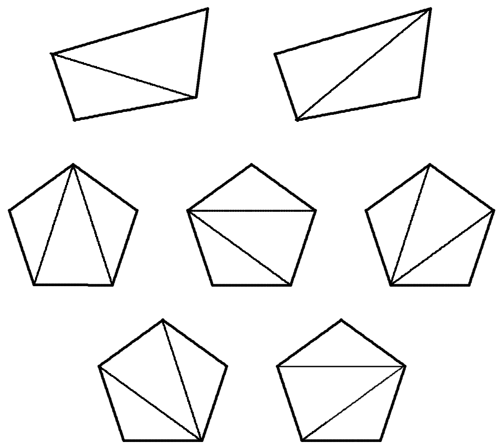
45. The Fibonacci ("rabbit") numbers are the sequence $(a_1 = 1)$, $1, 2, 3, 5, 8, 13, 21, 34, \ldots$, for which $a_{n+2} = a_{n+1} + a_n$ for any $n = 1, 2, \ldots$. Find the greatest common divisor of the numbers a_{100} and a_{99}.

46. Find the number of ways to cut a convex n-gon into triangles by cutting along non-intersecting diagonals. (These are the *Catalan* numbers, $c(n)$). For example, $c(4) = 2$, $c(5) = 5$, $c(6) = 14$. How can one find $c(10)$?



$a = 2, b = 2$

To Problems 42, 43



To Problem 46

47. There are n teams participating in a tournament. After each game, the losing team is knocked out of the tournament, and after $n-1$ games the team left is the winner of the tournament.

A schedule for the tournament may be written symbolically as (for example) $((a,(b,c)),d)$. This notation means that there are four teams participating. First b plays c, then the winner plays a, then the winner of this second game plays d.

How many possible schedules are there if there are 10 teams in the tournament?

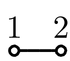
For 2 teams, we have only (a,b), and there is only one schedule.

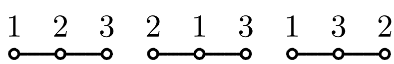
For 3 teams, the only possible schedules are $((a,b),c)$, or $((a,c),b)$, or $((b,c),a)$, and are 3 possible schedules.

For 4 teams we have 15 possible schedules:

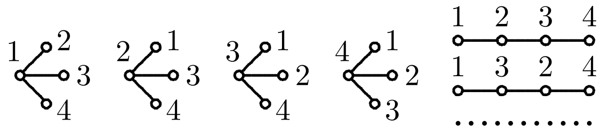
$$\begin{array}{llll}
(((a,b),c),d); & (((a,c),b),\ d); & (((a,d),b),c); & (((b,c),a),d); \\
(((b,d),a),c); & (((c,d),a),b); & (((a,b),d),c); & (((a,c),d),b); \\
(((a,d),c),b); & (((b,c),d),a); & (((b,d),c),a); & (((c,d),b),a);
\end{array}$$
$$((a,b),(c,d)); \quad ((a,c),\ (b,d)); \quad ((a,d),(b,c)).$$

48. We connect n points $1,2,\dots,n$ with $n-1$ segments to form a tree. How many different trees can we get? (Even the case $n=5$ is interesting!)

$n = 2$:  the number $= 1$;

$n = 3$:  the number $= 3$;

49. A permutation $(x_1, x_2, \dots, x_n)$ of the numbers $\{1, 2, \dots, n\}$ is called a *snake* (of length n) if $x_1 < x_2 > x_3 < x_4 > \cdots$.

$n = 4$: 

the number $= 16$.

Examples. $n = 2$, only $1 < 2$, the number $= 1$;

$n = 3$, $\left.\begin{array}{l} 1 < 3 > 2 \\ 2 < 3 > 1 \end{array}\right\}$, the number $= 2$;

$n = 4$, $\left.\begin{array}{l} 1 < 3 > 2 < 4 \\ 1 < 4 > 2 < 3 \\ 2 < 3 > 1 < 4 \\ 2 < 4 > 1 < 3 \\ 3 < 4 > 1 < 2 \end{array}\right\}$, the number $= 5$;

Find the number of snakes of length 10.

50. Let s_n denote the number of snakes of length n, so that

$$s_1 = 1,\ s_2 = 1,\ s_3 = 2,\ s_4 = 5,\ s_5 = 16,\ s_6 = 61.$$

Prove that the Taylor series for the tangent function is

$$\tan x = 1\frac{x^1}{1!} + 2\frac{x^3}{3!} + 16\frac{x^5}{5!} + \cdots = \sum_{k=1}^{\infty} s_{2k-1}\frac{x^{2k-1}}{(2k-1)!}.$$

51. Find the sum of the series

$$1 + 1\frac{x^2}{2!} + 5\frac{x^4}{4!} + 61\frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} s_{2k}\frac{x^{2k}}{(2k)!}.$$

52. For $s > 1$, prove the identity

$$\prod_{p=2}^{\infty} \frac{1}{1 - \dfrac{1}{p^s}} = \sum_{n+1}^{\infty} \frac{1}{n^s}.$$

(The product is over all prime numbers p, and the summation over all natural numbers n.)

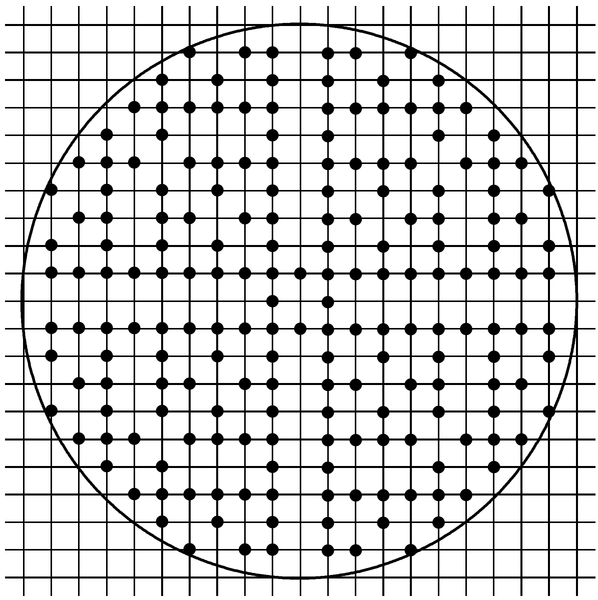
53. Find the sum of the series

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$\left(\text{Prove that it is equal to } \dfrac{\pi^2}{6}\text{, or approximately } \dfrac{3}{2}\right).$

54. Find the probability that the fraction $\dfrac{p}{q}$ is in lowest terms.

This probability is defined as follows: in the disk $p^2 + q^2 \leq R^2$, we count the number $N(R)$ of points with integer coordinates p and q not having a common divisor greater than 1. Then we take the limit of the ratio $N(R)/M(R)$, where $M(R)$ is the total number of integer points in the disk ($M \sim \pi R^2$).



$$\begin{aligned} N(10) &= 192 \\ M(10) &= 316 \\ N/M &= 192/316 \\ &\approx 0.6076 \end{aligned}$$

55. The sequence of Fibonacci numbers was defined in problem 45. Find the limit of the ratio a_{n+1}/a_n as n approaches infinity:

$$\frac{a_{n+1}}{a_n} = 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \ldots .$$

Answer: "The golden ratio", $\dfrac{\sqrt{5}+1}{2} \approx 1.618$.

This is the ratio of the sides of a postcard which stays similar to itself if we snip off a square whose side is the smaller side of the postcard.

How is the golden ratio related to a regular pentagon and a five-pointed star?

56. Calculate the value of the infinite continued fraction

$$1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}},$$

where $a_{2k} = 1$, $a_{2k+1} = 2$.

That is, find the limit as n approaches infinity of

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

57. Find the polynomials $y = \cos 3(\arccos x)$, $y = \cos 4(\arccos x)$, $y = \cos n(\arccos x)$, where $|x| \leq 1$.

58. Calculate the sum of the k^{th} powers of the n complex n^{th} roots of unity.

59. On the (x, y)−plane, draw the curves defined parametrically:

$$\{x = \cos 2t,\ y = \sin 3t\},\ \{x = t^3 - 3t,\ y = t^4 - 2t^2\}.$$

60. Calculate (with an error of not more than 10% of the answer)

$$\int_0^{2\pi} \sin^{100} x\ dx.$$

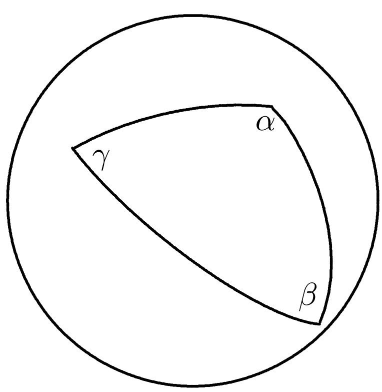
61. Calculate (with an error of not more than 10% of the answer)

$$\int_1^{10} x^x\ dx.$$

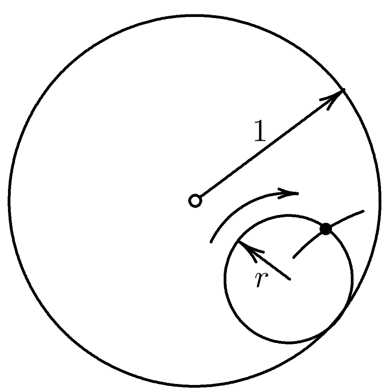
62. Find the area of a spherical triangle with angles (α, β, γ) on a sphere of radius 1. (The sides of such a triangle are great circles; that is, cross-sections of the sphere formed by planes passing through its center).

Answer: $S = \alpha + \beta + \gamma - \pi$. (For example, for a triangle with three right angles, $S = \pi/2$, that is, one-eighth of the total area of the sphere).

63. A circle of radius r rolls (without slipping) inside a circle of radius 1. Draw the whole trajectory of a point on the rolling circle (this trajectory is called a hypocycloid) for $r = 1/3$, $r = 1/4$ for $r = 1/n$, for $r = p/q$, and for $r = 1/2$.



To Problem 62



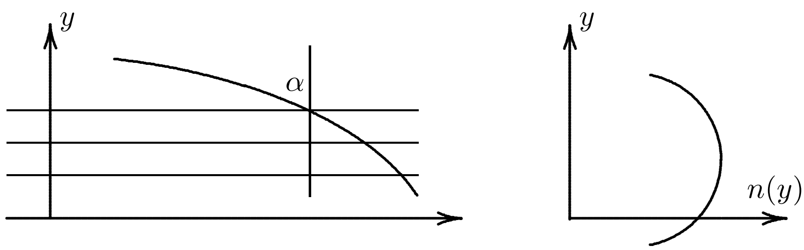
To Problem 63

64. In a class of n students, estimate the probability that two students have the same birthday. Is this a high probability? Or a low one?

Answer: (Very) high if the number of the pupils is (well) above some number n_0, (very) low if it is (well) below n_0, and what this n_0 actually is (when the probability $p \approx 1/2$) is what the problem is asking.

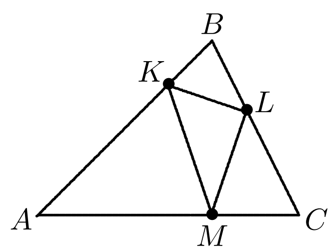
65. Snell's law states that the angle α made by a ray of light with the normal to layers of a stratified medium satisfies the equation $n(y)\sin\alpha =$ const, where $n(y)$ is the index of refraction of the layer at height y. (The quantity n is inversely proportional to the speed of light in the medium if we take its speed in a vacuum to be 1. In water $n = 4/3$).

Draw the rays forming the light's trajectories in the medium "air above a desert", where the index $n(y)$ has a maximum at a certain height. (See the diagram on the right.)



(A solution to this problem explains the phenomenon of mirages to those who understand how trajectories of rays emanating from objects are related to their images).

66. In an acute angled triangle ABC inscribe a triangle KLM of minimal perimeter (with its vertex K on AB, L on BC, M on CA).



Hint: The answer for non-acute angled triangles is not nearly as beautiful as the answer for acute angled triangles.

67. Calculate the average value of the function $1/r$ (where $r^2 = x^2 + y^2 + z^2$ is the distance to the origin from the point with coordinates (x, y, z)) on the sphere of radius R centred at the point (X, Y, Z).

Hint: The problem is related to Newton's law of gravitation and Coulomb's law in electricity. In the two-dimensional version of the problem, the given function should be replaced by $\ln r$, and the sphere by a circle.

68. The fact that $2^{10} = 1024 \approx 10^3$ implies that $\log_{10} 2 \approx 0.3$. Estimate by how much they differ, and calculate $\log_{10} 2$ to three decimal places.

69. Find $\log_{10} 4$, $\log_{10} 8$, $\log_{10} 5$, $\log_{10} 50$, $\log_{10} 32$, $\log_{10} 128$, $\log_{10} 125$, and $\log_{10} 64$ with the same precision.

70. Using the fact that $7^2 \approx 50$, find an approximate value for $\log_{10} 7$.

71. Knowing the values of $\log_{10} 64$ and $\log_{10} 7$, find $\log_{10} 9, \log_{10} 3, \log_{10} 6, \log_{10} 27$, and $\log_{10} 12$.

72. Using the fact that $\ln(1+x) \approx x$ (where ln means $\log_e$), find $\log_{10} e$ and $\ln 10$ from the relation[16]

$$\log_{10} a = \frac{\ln a}{\ln 10}$$

and from the values of $\log_{10} a$ computed earlier (for example, for $a = 128/125$, $a = 1024/1000$ and so on).

Solutions to Problems 67–71 will give us, after a half hour of computation, a table of four-digit logarithms of any numbers using products of numbers whose logarithms have been already found as points of support and the formula

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

for corrections. (This is how Newton compiled a table of 40-digit logarithms!).

73. Consider the sequence of powers of two: 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, ... Among the first twelve numbers, four have decimal numerals starting with 1, and none have decimal numerals starting with 7.

Prove that in the limit as $n \to \infty$ each digit will be met with as the first digit of the numbers 2^m, $0 \le m \le n$, with a certain average frequency: $p_1 \approx 30\%, p_2 \approx 18\%, \ldots, p_9 \approx 4\%$.

74. Verify the behavior of the first digits of powers of three: 1, 3, 9, 2, 8, 2, 7, ... Prove that, in the limit, here we also get certain frequencies and that the frequencies are same as for the powers of two. Find an exact formula for $p_1, \ldots,\ p_9$.

Hint: The first digit of a number x is determined by the fractional part of the number $\log_{10} x$. Therefore one has to consider the sequence of fractional parts of the numbers ma, where $a = \log_{10} 2$.

Prove that these fractional parts are uniformly distributed over the interval from 0 to 1: of the n fractional parts of the numbers ma, $0 \le m < n$,

[16]Euler's constant $e = 2.71828\cdots$ is defined as the limit of the sequence $\left(1 + \frac{1}{n}\right)^n$ as $n \to \infty$. It is equal to the sum of the series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$. It can also be defined by the given formula for $\ln(1+x)$: $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$.

a subinterval A will contain the quantity $k_n(A)$ such that as $n \to \infty$, $\lim(k_n(A)/n) =$ the length of the subinterval A.

75. Let $g : M \to M$ be a smooth map of a bounded domain M onto itself which is one-to-one and preserves areas (volumes in the multi-dimensional case) of domains.

Prove that in any neighborhood U of any point of M and for any N there exists a point x such that $g^T x$ is also in U for a certain integer $T > N$ (the "Recurrence Theorem").

76. Let M be the surface of a torus (with coordinates $\alpha \pmod{2\pi}, \beta \pmod{2\pi}$), and let $g(\alpha,\ \beta) = (\alpha + 1, \beta + \sqrt{2})$. Prove that for every point x of M the sequence of points $\{g^T(x)\}, T = 1, 2, \ldots$ is everywhere dense on the torus.

77. In the notation of problem 76, let

$$g(\alpha,\ \beta) = (2\alpha + \beta, \alpha + \beta) \pmod{2\pi}.$$

Prove that there is an everywhere dense subset of the torus consisting of periodic points x (that is, such that $g^{T(x)} = x$ for some integer $T(x) > 0$).

78. In the notation of Problem 77 prove that, for almost all points x of the torus, the sequence of points $\{g^T(x)\},\ T = 1, 2, \ldots$ is everywhere dense on the torus (that is, the points x without this property form a set of measure zero).

79. In Problems 76 and 78, prove that the sequence $\{g^T(x)\},\ T = 1, 2, \ldots$ is distributed over the torus uniformly: if a domain A contains $k_n(A)$ points out of the n points with $T = 1, 2, \ldots, n$, then

$$\lim_{n\to\infty} \frac{k_n(A)}{n} = \frac{\operatorname{mes} A}{\operatorname{mes} M}$$

(for example, for a Jordan measurable domain A of measure mes A).

Note to Problem 13. In posing this problem, I have tried to illustrate the difference in approaches to research by mathematicians and physicists in my invited paper in the journal "Advances in Physical Sciences" for the 2000 Centennial issue. My success far surpassed the goal I had in mind: the editors, unlike the preschool students on the experience with whom I based my plans, could not solve the problem. So they changed it to fit my answer of 4mm. in the following way: instead of "from the first page of the first volume to the last page of the second", they wrote "from the *last* page of the first volume to the *first* page of the second".

This true story is so implausible that I am including it here: the proof is the editors' version published by the journal.