Chapter 1

Molding a Math Circle

1.1 What is a mathematical circle?

On May 17, 2006, math department chairs from colleges and universities across the greater New York City area attended an organizational meeting held at the Courant Institute of Mathematical Sciences to take part in a mock math circle meeting and consider the possibility of starting one at their own institution. The event was organized by Mark Saul, a former administrator at the National Science Foundation and a long-time advocate of math circles. In a letter sent to departments announcing the meeting, Mark described math circles in this way:

> Mathematical circles are a form of outreach that bring mathematicians into direct contact with pre-college students. These students, and sometimes their teachers, meet with a mathematician or graduate student in an informal setting, after school or on weekends, to work on interesting problems or topics in mathematics. The goal is to get the students excited about the mathematics they are learning; to give them a setting that encourages them to become passionate about mathematics.

Defining a math circle

> Math circles can have a variety of styles. Some are very informal, with the learning proceeding through games or hands-on activities. Others are more traditional enrichment classes, but without formal examinations. Some have a strong emphasis on preparing for olympiad competitions; some avoid competition

> as much as possible. Models can use any combination of these techniques, depending on the audience, the mathematician, and the environment of the circle.

What all math circles have in common

> Athletes have sports teams through which to deepen their involvement with sports; math circles can play a similar role for kids who like to think. One feature all math circles have in common is that they are composed of students who enjoy learning mathematics, and the circle gives them a social context in which to do so.

The purpose of Circle-in-a-Box

The discussion, advice, and vignettes contained in these pages will convey a fairly complete picture of math circles. By the end most readers should be convinced that a math circle is a good idea for their institution, students, or child, depending upon their perspective. But that is not the true purpose of this document. More than just being informative, it is meant to function as a resource for individuals who are willing to take the leap of faith and actually initiate and maintain a math circle within their own community. These directors and coordinators, some of whom are featured in the "Circle Snapshots" sprinkled throughout the upcoming pages, form another common thread among all math circles. Each person has a unique organizational style, but they all share the belief that presenting beautiful mathematics to motivated students in an engaging manner is a worthwhile, important enterprise.

> *One feature all math circles have in common: they are composed of students who enjoy learning math, and the circle gives them a social context in which to do so.*

How do math circles begin?

There are as many compelling stories behind the launching of math circles as there are organizers. Perhaps a faculty member at a college or university owes their interest in mathematics to a circle they attended as a secondary student, and now wishes to provide a similar experience for students in their area. Since the math circle phenomenon is a relative newcomer to the United States, these professors tend to have arrived from a country with a rich tradition in math circles, such as Russia or Bulgaria. This was the case with Zvezdelina Stankova and the Berkeley Math Circle, for instance. Or perhaps a parent summons the courage to approach a local math department to forward the idea of partnering to establish a math circle, as occurred when Jennifer Jeffrey contacted Steven Krantz at the University of Washington, St. Louis. Bob and Ellen Kaplan, who both taught in the Boston area, became so tired of the negative attitude towards math ingrained in their students that they invited a group to meet in their living room to discover the true beauty of the subject. This informal gathering eventually grew to become The Math Circle.

A vibrant math circle can be a source of great inspiration to students and a rewarding enterprise for the mathematicians coordinating and leading them. However, just as with any sort of community, there is more to establishing a thriving math circle than meets the eye. Introducing students to exciting, accessible mathematics in a creative, engaging manner is certainly a key element. For this reason the entire second part of this handbook is devoted to laying out a wide variety of sample presentations, with a range of topics and difficulty levels, along with detailed presentation notes for the instructor and carefully developed problems and solutions. However, while solid mathematical content and delivery is certainly necessary for a successful math circle experience, it is not sufficient. Careful thought must also be given to the issues of how to attract and retain students, when and where to meet, how to structure the meeting time, whether to set up a web page for the circle, and more. Following a brief historical interlude, the upcoming sections will explore the various options available and consider how a number of successful math circles have addressed these issues.

> *While solid math content and delivery is certainly necessary for a successful math circle experience, it's not sufficient.*

Logistics behind a successful math circle

1.2 Origins

Mathematical enrichment activities in the United States have been around for at least thirty years, in the form of residential summer programs, math contests, and local school-based programs. The concept of a math circle, on the other hand, with its emphasis on convening professional mathematicians and secondary school students on a regular basis to solve problems, has appeared only within the past twelve years. This form of mathematical outreach made its way to the U.S. most directly from

Russia and Bulgaria, where it has been a fixture of their mathematical culture for decades. (The first ones appeared in Russia during the 1930's; they have existed in Bulgaria for a century.)

Tracing the roots of math circles

The tradition arrived with émigrés who had received their inspiration from math circles as teenagers. Many of them successfully climbed the academic ladder to secure positions within universities, and a few pioneers among them decided to initiate math circles within their communities to preserve the tradition which had been so pivotal in their own formation as mathematicians. The Mathematical Sciences Research Institute (MSRI) in Berkeley, California became involved at an early stage by supporting the Berkeley Math Circle. Not long after Steve Olson highlighted this math circle in his book *Countdown*, since a couple of members of the 2001 U.S. International Mathematical Olympiad (IMO) team attributed their success in part to the problem-solving sessions offered at Berkeley. In this and other ways math circles began to attract national attention as a means for encouraging students to enjoy, explore, and excel in mathematics.

> *Math circles function as an entry point to school-level math culture in Russia.*

The author was part of a delegation from the United States that recently visited Russia to observe the secondary school math culture in Moscow and St. Petersburg first-hand, including visits to a number of active math circles. These circles constitute only one piece of a wide array of programs aimed at identifying and cultivating mathematical talent amongst the Russian student population, at least those privileged enough to live near an urban center with a university or institute supporting such activities.

The role of math circles in Russia

Russian students grow up within an educational system that promotes problem solving (as opposed to computation or textbook exercises) to a much greater degree than in the United States. Therefore it is a natural step for these students to begin taking olympiads and attending math circles during their middle school years. (In contrast to a math contest, an olympiad typically asks students for an explanation rather than a computation. These olympiads are offered at all levels and geographic scales; the most prestigious is the All-Russian Olympiad which helps to determine their IMO team. It is the Russian analogue of the USA Math Olympiad.) The olympiads and circles feed into one another and serve as the entry point to the school-level mathematical culture in Russia. From there the stronger students might be admitted to one of several "specialized schools" with advanced math classes taught by teachers whose mathematical pedigree is comparable to those at the best American high schools. Students who wish to pursue mathematics may also attend a selection of summer programs and participate in various special events such as "math battles" or "math fests."

It is illuminating to contrast this system with that currently in place in the United States. The most notable difference concerns the general perception of mathematics as a domain for investigation versus a tool for science or business. This is due in large part to the significantly higher level of interaction between research mathematicians and secondary school students in both Russia and Bulgaria. It is not uncommon for graduates with degrees in mathematics to enter the teaching work force as well as to continue active research at some level. It is also standard for well-known mathematical figures to join the staff at summer programs. The Moscow Center for Continuous Mathematics Education, which coordinates a wide array of activities for school-level students, is housed in the same building as the Independent University of Moscow and has ties with Moscow State University as well. Most striking, perhaps, is the incredible level of involvement of a considerable number of undergraduate and graduate students who spend hours each week helping out at math circle meetings, grading olympiads, or otherwise supporting the effort. Because of their faithfulness in repaying their debt to the prior generation of university students, kids attending math circles receive an unparalleled amount of personal attention. It is standard operating procedure to devote the bulk of a meeting to individual discussions with mentors, in which participants present their solutions to problems which had been distributed during the previous session.

Connections between schools and universities

At the same time, Russian mathematicians working to encourage students with an interest in math face many of the same challenges that their American counterparts do. There are state-mandated multiple choice tests for all students intended to guarantee minimal competency in math which divert attention from high performers. Those who conduct mathematical research often find themselves at odds with those who study mathematics education. Many students who would benefit greatly from attending math circles are geographically out of reach. Time and

Common challenges

Circle Snapshots

Name: The San Francisco Math Circle

Location: San Francisco State University

Director: Paul Zeitz

Email: `zeitz@usfca.edu`

Meeting time: Mondays 4:30–5:30pm

Web site: `www.sfmathcircle.org`

Paul Zeitz has felt called to provide mathematical opportunities to "unenriched kids" for a long time, due in no small part to the fact that he grew up fatherless in a gritty New York neighborhood. As a co-director of the Bay Area Math Olympiad, he was mildly distressed to notice that none of the participants (much less high scorers) were located in San Francisco. So when MSRI approached Paul the winter before his sabbatical year with an idea for presenting exciting mathematics to city kids and teachers, Paul was eager to help head up the new program.

The pivotal idea about which the San Francisco Math Circle revolves is the premise that the most effective way to reach these kids is to set up an excellent program for teachers and compensate them for bringing their students along. Teachers enjoy a quality mathematical experience, receive continuing education credit for regular participation, and earn $100 per week for transporting their students to the event. Paul also asks his presenters to commit to four or five weeks and chooses a common theme which all the groups investigate during that time. His philosophy seems to be working—on a typical Monday around fifteen teachers bring a diverse crowd of students, upwards of eighty kids each week.

Several factors have made this program as successful as it has become. According to Paul, finding their present location at SFSU was one of the keys, since this site is much more accessible to his target population. Paul notes that Marianne Smith deserves a substantial amount of the credit for helping to put the organizers in touch with a group of dedicated teachers and key administrators during the months leading up to their first session. Finally, none of this would be possible without significant financial backing, which was discovered through and is managed by the Mathematical Sciences Research Institute.

funding for worthwhile projects is often in short supply. But it is heartening to note that because of these similarities each nation can learn from successful approaches discovered and employed by the other.

1.3 Knowing your audience

By far the most important matter that the organizer of a new math circle must settle concerns its intended participants, as this decision will inform choices to be made on all the other issues raised in the upcoming sections. The arrangement that naturally springs to mind, and hence the most common model, is to set up a math circle targeting motivated students interested in math within a certain geographical area. In this case the common denominator among members of the audience is each individual's (or their parent's) knowledge of and desire to attend the circle, independent of school affiliation or other background. Although this is likely to be the appropriate choice in most cases, there are other possible models for math circles that target teachers or underserved populations. While they are not addressed separately here, information pertaining to these models appears throughout the text. In particular, the San Francisco Math Circle snapshot and the sample grant proposal in the appendices contain helpful ideas regarding math circles for these intended audiences.

Different models for math circles

The Math Circle in Boston reaches students as young as four years old.

For most new coordinators, the first truly substantial decision concerns the level at which to pitch the math circle. Given the organization of grades in the United States, the natural options to consider are the middle school level, high school level, one size fits all, or two or more levels offered simultaneously. (Although The Math Circle in Boston reaches students as young as four years old.) The final option is more ambitious than is advisable for a new circle, but it is a healthy development among circles which outgrow their original charters. In terms of content, material intended for middle school students may assume basic algebra such as factoring and the quadratic equation, techniques for solving simple word problems, standard area and volume formulas from geometry, sequences or patterns, and similar topics.

Topics for middle and high school students

High school students will have also been exposed to Euclidean geometry, polynomials, exponential and logarithmic functions, trigonometry, elementary counting techniques, and other precalculus topics. Therefore middle school students would certainly enjoy presentations introducing graph theory, basic probability, iterated functions, elementary number theory, and so on. However, more than likely they would become quickly lost if this material were presupposed. Topics that are better suited for a high school group might include combinatorial proofs of Fibonacci identities, Gaussian integers and sums of two squares, inversion in Euclidean geometry, complementary sequences, and a host of other exciting mathematics. Further thoughts on selecting topics are assembled in the chapter on leading math circles.

> *There is a force that has the effect of lowering the average age of a math circle over the years.*

The difference between middle and high school students

The range of mathematics available to high school students, as opposed to middle school students, turns out to be disproportionately broader than a few extra grade levels of mastered curriculum might suggest. There is no denying that older kids have a greater base of knowledge, which permits a wider variety of potential topics. But there is another more significant sociological factor at work. If a high schooler attends a math circle it is usually because that student has answered the question, “Am I interested in and good at math?” in the affirmative, has independently studied advanced mathematics, and is likely to be substantially beyond the average high school student in terms of mathematical experience. On the other hand, late elementary and middle school students (and their parents) are usually in the midst of exploring this question, and attendance at the local math circle can be part of the process of finding the answer. These students have not had nearly as many years in which they knew they wanted to focus on mathematical thinking. The point is, a high school group is usually further above grade level in mathematical experience, on average, than a middle school group. This fact should not preclude the presentation of challenging material that will stretch middle school students’ minds; presumably they are attending the circle because they are interested in discovering new mathematical horizons, even if they do not completely understand the material on the first pass. But it is a good idea to keep this distinction in mind.

The rich variety of interesting topics available at the high school level would seem to make this age group far more desirable, especially to a college or university math department. Therefore it may come as a surprise to learn that there are very few math circles which primarily serve high school students. (The Mobile Math Circle is one of the exceptions.) There is a counterintuitive force at work that has the effect of lowering the average age of a math circle over the years. This phenomenon will be discussed in more detail when dealing with the issue of retaining students, but it is important to note at this stage that a math circle which intends to target older students should be prepared to actively resist this tendency. For the same reason, organizers who decide to make their circle accessible to all age levels should commence such an undertaking with realistic expectations; practically every circle that begins in this fashion metamorphoses into a middle school math circle soon after.

> *"The [Mobile] Math Circle has been by far the most effective outreach activity we have ever undertaken."*

The Fountain of Youth effect

1.4 Location, location, location

Selecting a suitable meeting site is the next major decision to be made. Faculty members at a college or university will probably opt to host the circle in a classroom or auditorium regularly used by their math department. For secondary school teachers and parents the decision is less obvious. Although it may require more legwork, finding a math department at a local institution that is willing to sponsor and host the math circle has several strong advantages over running it out of a school, workplace, or home. For starters, a college or university provides a neutral, well-known, scholarly setting for the pursuit of mathematics. Besides, students tend to be excited by the prospect of spending time on a university campus. When the Peninsula Math Circle relocated to Stanford University after one semester of operation, its enrollment immediately doubled. Furthermore, corporations or private schools run the risk of having their mathematical outreach misconstrued as a recruitment effort. (Unless, of course, this is the intent.) Departmental sponsorship has the added advantage that finding speakers, which is one of the main challenges

Choosing a site for math circle meetings in operating a math circle, becomes much more feasible. At the end of the day there may be other constraints which make this option less appealing, but it is usually worth investigating.

Obtaining the support of a local math department

Here are a few thoughts for organizers without a university affiliation who wish to enlist the support of math faculty at a nearby institution. If possible, avoid introducing new projects to faculty members during the summer, when they are typically at conferences, involved in summer programs, engaged in research, or writing their books; or in the fall, when they have their hands full with the start of classes and other responsibilities. One is more likely to receive a warm reception in the winter or spring when there is time to plan ahead for a new departmental activity. Secondly, contact the chair of the department; this person should be kept apprised of the math circle and will know who in the department is most likely to be willing to help out. Mention the name of another person on the faculty if someone else is already interested in coordinating the math circle. Finally, mention that such a project would be beneficial to the department in addition to providing an exciting mathematical opportunity to students in the community. Dan Silver at the University of South Alabama (USA) stated that, "The [Mobile] Math Circle has been by far the most effective outreach activity we have ever undertaken." Part of the original motivation for their math circle was to draw students into the math major at USA; the strategy worked even better than they had hoped. It also had the effect of integrating the mathematical community in their region, bringing high school, college, and graduate students together with math faculty.

Benefits of a math circle to a department

> *It goes without saying that Friday night is not desirable, but Monday night works better than one would expect.*

Math circles at high schools

Secondary school teachers with the right combination of mathematical background, enthusiasm, and access to potential speakers have also managed to successfully conduct math circles at their schools. For instance, a math circle ran at Henry M. Gunn High School in Palo Alto, CA for many years. Making the event welcoming to students outside the school is more of a challenge in this case. Finding qualified people to lead the sessions can also be more difficult. Therefore these groups are often composed solely of students from within the school who spend time preparing for math contests or together poring over books containing mathematical activities or advanced topics. As such they are more aptly described as math clubs than math circles. However, exceptions do exist; the St. Mark's Institute of Mathematics is a good illustration.

1.5 Setting the schedule

The factors that will influence the choice of times and dates to meet vary so widely from one math circle to the next that it would be impractical to attempt to include them all. Each organizer will need to make allowances for room and speaker availability (don't compete with the Tuesday afternoon colloquium), local traditions and holidays (Wednesday night is out in the Bible Belt), traffic patterns (nobody would make it to campus on time before 6:30 on weekdays), and a myriad of other factors. However, there are still general principles that can be applied in a range of settings.

Weekday versus weekend meeting times

For starters, the age of the students attending the circle makes a difference: middle school students can squeeze a late afternoon time slot into their weekday schedules more easily than high schoolers, who have a greater number of demands placed on their after-school time. Both age groups can usually manage an early evening meeting time, although this may again be asking more of high school students given their typically larger homework load. It goes without saying that Friday night is not desirable, but Monday night works better than one would at first expect. Gathering on a weekend allows students from a wider geographic area to attend. Thus Saturday morning or Sunday afternoon have proven to be a viable alternative for many groups like the Boston Math Circle. (The official title of the latter group is simply "The Math Circle," but the modifier "Boston" will be inserted from here on to avoid confusion.) A weekend time is especially well-suited for larger groups such as the San Diego Math Circle.

> *It is preferable to shorten the year to allow regular meetings rather than to thin out a schedule with biweekly math circles.*

The intended mode of transport will also play an important role in determining a meeting time. The San Francisco Math Circle provides programs for both teachers and students; since these teachers drive their students to and from the circle a late

Circle Snapshots

Name: St. Mark's Institute of Mathematics

Location: St. Mark's School, MA

Director: Jim Tanton

Email: `mathinstitute@stmarksschool.org`

Meeting time: Mondays 4:30–6:30pm

Web site: `www.stmarksschool.org/academics/mathinstitute.aspx`

Jim Tanton hales from Australia. He completed his doctoral work at Princeton and began heading down the academic track before his passion for teaching diverted him to spend several years working with the Kaplans at the Boston Math Circle. When St. Mark's School in Southborough, MA invited him to join their faculty and found an Institute of Mathematics, Jim was delighted. The institute, whose mission is to engage in mathematical outreach at all levels, began operation in the fall of 2004.

True to its mission statement, the institute offers a wide variety of mathematical activities and programs. Jim leads two hour-long math sessions on Monday afternoons. The middle school students appear at 4:30, followed by a high school group. The circle is open to all kids in the area; more often than not the local students outnumber those from St. Mark's. Jim also conducts five core courses for teachers who are working towards a Master of Education degree with a concentration in math. The classes, which are offered through Northeastern University, meet at the Institute on Saturdays or during the summer and provide the primary source of funds for the Institute's activities.

Besides holding forth on his favorite mathematical topics, Jim also loves to write about mathematics. Several years ago he decided to publish a monthly newsletter containing interesting problems, engaging exposition, and ideas for further research, all centered about a common theme. This lively document, typically filling the front and back of one sheet of paper (a different color each month), was originally mailed out to math departments in the greater Boston area. The circulation has now increased to around six-hundred addresses, many of them students. To this day nobody has ever requested to be taken off the mailing list.

afternoon time is essentially the only option available. A math circle targeting an urban population that would travel via public transportation might also be better off in the afternoon, when service is more frequent. On the other hand, parents are the primary chauffeurs to the San Jose and Stanford Math Circles, which meet on Wednesday evenings and Sunday afternoons, respectively. To some extent, families will organize their schedules around a math circle they strongly wish to attend; often the best plan is to pick a feasible time and stick with it for a year, knowing that it is impossible to please everyone.

Transportation is a factor

Recruiting a sufficient number of good speakers to lead the circles is the greatest yearly challenge organizers face.

Most organizers make very similar choices with respect to meeting dates. It makes sense to slate the initial meeting a week or two after Labor Day; by this time students have begun to establish a weekly routine but have not yet become too busy to consider a new activity. This time frame also gives coordinators the chance to advertise the circle. Universities on a quarter system may opt for an October start, but should probably begin as early as their academic schedule permits in light of the previous considerations. There is time for a couple of meetings after Thanksgiving, although attendance usually dwindles. Following winter break the default would be to continue meeting until final exams begin to loom for either those attending the circle or those hosting it, which typically occurs in late April or early May. However, there may be good reason to draw the year to a close slightly earlier, such as difficulty in finding enough good speakers to support a full year's worth of meetings, or simply a desire to avoid an overly ambitious schedule the first year. In early April the Stanford Math Circle segues into practice sessions for the American Regions Math League (ARML) team, which take place at the same time and location.

The calendar

Coordinators of most existing math circles favor weekly meetings, for good reason. There is much to be gained from holding a regular math circle—the event becomes part of a weekly routine and it is possible to establish a sense of continuity. A student who misses one biweekly meeting (even if for a good reason!) goes three weeks without attending the math circle, at which point she decides that she can live without it. One exception to this rule occurs when a middle school circle gathering every other

How often should a math circle meet?

week is associated with a companion high school circle meeting more regularly, so that students have a fall-back option during the off weeks. But in general it is preferable to shorten the year in favor of regular meetings rather than to thin out a schedule with biweekly math circles.

Lastly, almost all math circle coordinators make the same choice with regard to the length and format of a meeting. While an hour is sufficient for presenting significant mathematics in a lecture format, it is barely enough time to allow students to make substantial headway in exploring a new topic for themselves and to actually work on problems in the process. On the other hand, two hours begins to approach the limit of a group's attention span, even with a short break in the middle. Besides, it would be asking a lot of a family to devote more time than this to any single activity. Therefore most sessions last from an hour and a half to two hours in duration. Announcements, mathematical tidbits, or speaker introductions usually occupy the first five to ten minutes while those caught in traffic arrive. The presentation for the day follows; the style of delivery should be so interactive that there is no need to reserve time for questions at the end. Instead, save a few minutes for snacks, or provide a ten minute break during longer sessions.

One of the most popular presentations made at the Stanford Math Circle last year was given by a local high school math teacher.

How long should a meeting last?

1.6 Filling the schedule

As all veteran math circle coordinators can relate, recruiting a sufficient number of good speakers to lead the circles is the greatest challenge they face from year to year. Their selection is crucial, for the person presenting the mathematics will determine whether the participants eagerly explore or gradually disengage from the day's topic. Suffice it to say that the job calls for an understanding of interesting, accessible mathematical topics that are not part of the standard secondary curriculum; knowledge held by individuals who typically pursue careers in academia, software engineering, national security, or other fields requiring specialized math skills. On the other hand, this knowledge could be squandered if it is not presented in an engaging, interactive,

The greatest challenge to the coordinator creative manner; much as a favorite middle or high school math teacher would have done. Paul Zeitz, the director of the San Francisco Math Circle, has this cautionary advice to offer:

> Don't make the assumption that knowing cool math will immediately translate into an exciting, compelling mathematical experience for secondary school students! It is harder to break with the lecture style of presentation than most people realize, and delivery makes a difference.

This being said, there are a few guidelines that will help ensure a solid line-up of speakers for any math circle. To begin, start looking around early—the best people are usually the busiest. Look for people that fall into at least one of the categories mentioned above, then provide resources to give support in the other category. For example, college or university faculty members often have interesting topics at their fingertips. However, they are accustomed to lecturing to advanced students and may not have as much experience conducting an interactive, open-ended exploration at the secondary school level. Sending out a friendly "what to expect" letter a week or two in advance, which includes a description of the style and content employed at a typical math circle meeting, will go a long way towards bridging the gap and will probably be much appreciated. (A sample email message is contained in the appendices.) It is also a good idea to ask speakers to prepare a handout ahead of time with a variety of problems related to the presentation. This will help keep students occupied and engaged during the session, and will also provide a gauge of the difficulty level of the topic, in case adjustments need to be made given the background of the audience. Above all, speakers unfamiliar with the math circle philosophy will benefit from seeing one in action, so invite new instructors to visit the circle a week or two in advance of their debut. This practice can become standard protocol: at the Utah Math Circle all first-time speakers are required to attend a math circle before leading a session.

Finding good speakers for a math circle

While substantial mathematical content should constitute the staple of any math circle diet, there is certainly latitude for incorporating related subjects.

Tips for inviting college level speakers

Candidates for math circle speakers

Professors are certainly not the only ones with the requisite mathematical background. Graduate students, high school teachers, and a wide variety of professionals can all make excellent candidates for leading math circles. One of the most popular presentations at the Stanford Math Circle last year was given by a local private school math teacher. In theory undergraduates and middle school teachers could also be included in this list, but these two groups might be uncomfortable at the helm. Undergraduates may be too close in age to their audience, and will probably lack speaking experience. Middle school teachers would be more effective than they might suspect, but may not consider themselves sufficiently advanced in math to feel comfortable leading a mathematical enrichment activity.

Qualities of good speakers

Regardless of where one looks, the universally accepted method for tracking down math experts can be summarized as: keep an ear to the ground and persistently ask around. It makes sense to start with local colleges or universities, but look further afield also. Keep in mind that the most important qualities for a speaker to possess are a dynamic, engaging presentation style, a love for and general competency in mathematics, and some familiarity with the target age group. If there is money available to support a guest's travel and accommodations, consider inviting a more well-known mathematician to speak at the circle session. Anticipating such a visit can provide focus and generate enthusiasm among regular math circle participants. It might work well to arrange for this guest to also give a department colloquium; the trip becomes twice as worthwhile for the speaker and other funds may become available for covering the visit. Naturally the director of the circle should speak frequently as well, without letting the responsibility of preparing for sessions turn into an unwelcome burden.

> *The percentage of teachers who disregard math circle flyers is often too high to make this means of contacting students viable for organizers.*

Most individuals who enjoy mathematics will already have their own list of favorite problems which they would delight in discussing. However, some potential speakers may be unsure as to which topics would be best to present since they do not have regular contact with pre-college students or may not have a working knowledge of mathematics outside the standard secondary curriculum that is both accessible and well-suited for exploration. Look no further—the chapter on leading a math circle includes a long list of possible topics along with suggestions on how to present them. Furthermore, the second part of this handbook contains a variety of ready-made math circle sessions that can be used directly or adapted for a particular audience. And while substantial mathematical content should constitute the staple of any math circle diet, there is certainly latitude for incorporating related subjects into the schedule. Thus students might enjoy learning about mathematics as it is applied to computer graphics, or sailboat design, or survey data analysis, for example.

Resources for speakers

Finally, there is the question of how often individuals will visit a particular math circle. Should a speaker come just once? This approach places less of a burden on the speaker and ensures a greater variety of topics during the year. However, this practice also makes it more challenging to fill up a schedule and gives the math circle a bit of a guest lecture series flavor. (There are ways to counteract this effect, described in the section on retaining students.) Should a speaker come twice in a row? Fewer people may be willing to commit to two appearances, but the advantages gained include an increased sense of continuity between meetings, the ability for a speaker to build on material from the previous week, and a far easier time slating enough presentations for the year. The majority of math circle coordinators attempt to book speakers for at least a couple of visits, particularly those who are known to be effective. At the other end of the spectrum, some directors arrange for instructors to conduct an entire block of math circles lasting a month or more in length. This approach allows students to become very familiar with the speaker, promotes regular attendance, and permits a more thorough treatment of a subject by the instructor. It also calls for a different model of math circle, in which speakers are paid for their time through the support of outside funding. The math circle might also choose to charge tuition to help guarantee the regular presence of excellent instructors; this issue will be addressed in more detail in an upcoming section on funding. Of course, the math circle coordinator might wish to lead a fair number of sessions as well. This has the desirable effect of allowing the coordinator to become more familiar with the students. However, there is a balance to be struck here; a variety of presentations is good for the health of the circle.

How many times should a speaker visit?

Circle Snapshots

Name: The Math Circle

Location: various Boston area sites

Director: Robert and Ellen Kaplan

Email: kaplan@math.harvard.edu

Meeting time: various times

Web site: www.themathcircle.org

The following excerpt, slightly updated, is taken from the history given at The Math Circle web site:

Disturbed by the poor quality and low level of math education in the country, three of us (Bob and Ellen Kaplan, and our colleague Tomás Guillermo) began The Math Circle in September 1994. We rented space on Saturday mornings in a local church and word of mouth alone brought us twenty-nine students for that first (ten session) semester. Scholarships were offered when needed, and we almost covered our expenses.

The Saturday format we set up then continues still—but now on Sunday mornings to avoid soccer conflicts. The students are divided into two groups (roughly by age) and now Gordon Ritter and Sam Lichtenstein teach a class from 9:15 to 10:00. After a fifteen minute break for juice and cookies, we change places and teach a different group from 10:15 to 11:00. In the last hour we all listen to an invited speaker or work on a joint problem.

In the second semester of our first year, Northeastern University (through Andrei Zelevinsky) offered us free—and much larger—quarters; and demand for a younger class led to a single session on Thursday afternoons, in a room offered us free by Harvard (thanks to Danny Goroff). Enrollment rose to thirty-four.

We began our second year at Northeastern and Harvard with thirty-eight students (some coming via the article about The Math Circle, which had just appeared in the AMS Notices), so we added a Harvard graduate student to our staff, setting a pattern which has allowed for our expansion to two hundred five students. The Sunday classes have, of necessity, stayed about the same size, but the weekday sessions for five to eleven year olds have proliferated: this year there are nine classes meeting at Harvard, on Tuesday and Wednesday late afternoons.

1.7 Getting the word out

Scheduling speakers may be the most challenging aspect of coordinating a math circle, but the task of successfully advertising the circle to students runs a close second. One of the most effective ways of notifying potential participants, of course, is to send a message directly to the students themselves. Various organizations that offer math or science programs for secondary schools might be willing to contact the students on their email lists on the circle's behalf. Naturally it is a good idea to write out a paragraph or two introducing the math circle which can be used within their message. This approach has the added advantage that the math circle is implicitly endorsed by an organization with which the students have (hopefully) already had a positive experience. Students heard about the Stanford Math Circle in this fashion, when the local ARML coach and the Art of Problem Solving web site both sent out messages to students in the area less than two weeks before the initial meeting. Nearly fifty students (and a substantial number of parents) appeared the first weekend, forcing the event to move to a larger auditorium at the last minute.

Contacting students directly

Other methods for contacting students can be loosely classified according to the messenger. The most obvious choice of person to deliver a flyer or make an announcement would seem to be a math teacher, thereby ensuring that the news reached students at the appropriate grade level and geographic region. This strategy can work, and is in fact one of the best ways to reach "unenriched" or minority populations. However, results will vary dramatically depending on the amount of initiative displayed by the various teachers. As part of their publicity effort, the Mobile Math Circle mailed flyers to all teachers at nearby public high schools to announce their circle. They proceeded to discover that a disproportionate number of kids all came due to the efforts of one particularly enthusiastic math teacher. In the end, the percentage of teachers who disregard math circle flyers is often too high to make this means of contacting students viable for organizers.

Teachers as messengers

One teacher allowed students participating in the weekend math circle to skip the assignment due Monday morning.

As something of an extreme example, consider the cautionary tale of the University of Birmingham, whose math department took it upon themselves to provide an uplifting mathematical experience for boys and girls attending the neighborhood city schools. A few organizers put together the entire math circle package, complete with a schedule of speakers, space to conduct Saturday morning events, even funding to provide transportation to and from local schools each weekend. Unfortunately, nobody thought to get the teachers on board. When a comprehensive mailing was sent out announcing the new math circle, the organizers received exactly one response, and the program folded before it began.

The missing link

When a teacher can be found who is willing to steer students towards the math circle, it is important to impress upon them the type of student who would benefit most from a math circle. In particular, there is much lower correlation between grades in school and math circle appreciation than most teachers would expect. Students should attend a math circle out of a innate love for the subject: because they are fascinated by the clever patterns among Fibonacci numbers, because they can't wait to see how number theory is used to enable secure transactions over the internet, or simply because they love to solve problems. Some of these kids are also A students, but some are not. To put it another way, some students earn top grades in math because they are academically competitive or because they have become quite adroit at solving textbook exercises. These facets of classroom education are either downplayed or nonexistent at a math circle, making the environment much less appealing to these types of students. As an illustration, one math circle encouraged teachers to offer extra credit to students who attended. Not surprisingly, this ploy attracted the wrong crowd, and was abandoned some time later. On the other hand, a teacher from another area allowed students attending the weekend math circle to skip the assignment due Monday morning. Since it was simpler for the academically minded kids to just polish off their homework questions, only those truly interested in the math circle consistently accepted her offer. To recapitulate, it is more important for a student to be interested in math than accomplished at math.

Identifying students who would enjoy a math circle

Incentives for attendance

> *When it comes to penetrating bureaucracy, finding the right person makes all the difference.*

The return on advertising investment drops even further when attempting to reach students and teachers through school administrations. Experience has shown that most school systems simply ignore appeals to advertise math circles. Given the number of organizations that would benefit from contacting potential customers in this manner, one can hardly blame them. The exception to this rule occurs when a math circle is able to gain the support of an individual with connections to those who oversee or promote math enrichment within the school district. The San Francisco Math Circle illustrates a perfect example of this sort of phenomenon. They attempted to reach math teachers via central school offices for weeks to absolutely no avail. Fortunately, they were able to contact an independent consultant who helped set up meetings with math department chairs in the area. To the relief of the organizers, interest in this new math circle for students and teachers quickly sprang up, resulting in a much more substantial turnout than originally anticipated. When it comes to penetrating bureaucracy, finding the right person makes all the difference.

Navigating bureaucracy

There are practically as many other creative means of disseminating information as there are math circles. Ideas that have been tried, or at least considered, include the following:

- Make an announcement in the weekly college newsletter sent out to all employees. This tactic can net a substantial number of faculty kids.

- Have hand-picked graduate students (or faculty members, better yet) give guest presentations at math clubs in local secondary schools, then distribute information about the math circle.

- Send a press release to local newspapers and TV or radio stations. One math circle landed a spot on the morning news this way. Another received quite a few phone calls from interested parents after a widely-circulated paper ran a story on math circles.

- Find a parent who is willing to speak directly with teachers to advertise the math circle, since teachers can be intimidated by and hence dismissive of a university-based program.

- Encourage supportive parents to spread the word among families in their after school programs, churches, workplaces, or anywhere else.

- Set up a direct mailing of flyers to all residential addresses in the area announcing the circle. This approach requires some financial outlay, of course, but is hard to beat in terms of sheer impact.

The biggest hurdle to overcome is convincing students to attend once. After they have found time in their schedule, navigated parking lots, located the meeting room, and met a few of their mathematical peers, they will realize that math circles are truly wonderful events and will want to return regularly.

Chapter 3

Sustaining a Math Circle

Having given thought to the issues related to the birth of a new math circle, the time has come to consider its nurture and development. One of the foremost challenges that an organizer faces during the course of a year is to reach and maintain a comfortable size. Some drop-off in attendance during the first few weeks is to be expected, but ideally the number of students will not fluctuate much beyond that. The organizer will also address other matters as the year progresses, such as how to propel especially motivated participants further along in mathematics, or how to handle the weekly maintenance of the circle most efficiently. However, the latter topics hinge upon sustaining the level of interest among students that brought them to the circle in the first place, so we begin with this issue.

Reaching a suitable size

3.1 Retaining students

Having expended considerable time and energy to create an opportunity for students to broaden their mathematical horizons, it is enormously affirming to see a solid group of kids taking advantage of this opportunity week after week. Assuming that there is a sufficiently large population of students in the area who are genuinely interested in what a math circle has to offer, it is not hard to establish a core group of faithful participants. There are a few common sense principles to observe which will help to ensure that attrition never becomes an issue. (It can be one of the more discouraging issues—to be avoided if possible.) A few of these principles have already been touched upon, like choosing

Maintaining a suitable size

an appropriate meeting time based on the intended audience or asking families to invest in the circle financially by encourage donations or charging a nominal tuition. The remaining suggestions are all related to activities that occur at the math circle itself as opposed to decisions made during the planning phase.

Conjuring up community

In the absence of extra incentives, students will attend an event such as a math circle if they (or their parents) feel that they are part of a worthwhile activity and that their presence is significant. The quality of the mathematical content and its presentation is the subject of the second part of this handbook; for now let us consider ways in which students can be made to feel that they are meaningfully connected to the math circle. The task is to turn a collection of students of various ages from a number of schools with differing mathematical backgrounds into a cohesive community. This process is more of an art than a science; as such, it is not necessarily the forte of every mathematician. However, the following ideas can be implemented by any organizer who wishes to build a thriving math circle.

> *The more a participant feels known by the coordinator and by their peers, the more likely they are to attend regularly.*

Establishing continuity

One of the most powerful means for accomplishing this goal is to establish a thread of continuity from one meeting to the next. Adhering to a uniform format, location, and meeting time is essential. If possible, the coordinator should attend every meeting in order to welcome students, make announcements, and introduce speakers. The regular presence of the person in charge helps to make the circle seem more familiar and accessible to students. Better yet, arrange to have speakers come for two or more weeks in a row, thereby halving the number of new faces that students must contend with and allowing students and instructors to become acquainted. (One should be prepared to offer non-trivial honoraria in this case.) Another effective strategy is to involve students in endeavors that span the course of several weeks or an entire semester. For example, at the first meeting students could be given a "Math Circle Challenge" consisting of a sheet of problems of varying levels of difficulty. The first ten minutes of each session could be devoted to presenting solutions, with the promise of a prize for the entire group on the final meeting of the semester commensurate with their collective progress.

It would be hard to overstate the impact that is made by an organizer who takes the time to greet students by name as they arrive, asks them if they have come across any interesting mathematics lately, or lends them an interesting puzzle to play with. The more that a participant feels known by the coordinator and by their peers, the more likely they are to attend regularly. Name tags for everyone at the first several meetings might help, although this approach is more likely to be effective at the middle school level. Use the initial moments of the meeting to run activities that help students connect with one another. For instance, one could instruct everyone to write down a secret positive integer from 1 to 20, with the caveat that they should select a number which they think that nobody else has chosen. (Adjust the range as necessary based on the size of the group.) Those who perform the task successfully win chocolates, those who don't will have at least created a small bond with those other members of the circle who read their minds and spoiled their chance for a treat. By the way, this simple game quickly leads to some interesting mathematical questions in probability and expected value.

Help students to feel plugged in

An ice-breaker

Along the same lines, incorporate plenty of time for mathematical and social interaction. Encourage speakers to set aside time for kids to work together on problems. Many topics can be motivated by games or numerical investigations upon which students can embark in groups of two or three. Have students present ideas and solutions on the board, thus giving them a stake in the proceedings. Schedule time during or after the session for snacks and chatting. Plan a holiday party (with a mathematical theme, of course) or an ice-cream social for the second half of the final meeting of each term. The group could also arrange to attend an event of mathematical significance together. While the social aspect of a math circle should certainly not become its focal point, a healthy amount of interaction is important to its well-being.

> *Students often need an extra nudge to attend an event they would sincerely enjoy and benefit from once they arrived.*

Incorporating some social interaction

Incorporating special events into the calendar also serves as an effective antidote to attrition. Math circles in the San Francisco Bay Area all turn their attention to promoting the Bay Area Math Olympiad (BAMO) in late February and helping students to prepare for this intense proof-oriented competition. Everyone subsequently attends a lavish awards ceremony hosted by MSRI during the second weekend of March to hear a fabulous guest lecture, honor the high scoring students, and enjoy a nice lunch.

Special events are beneficial

Most circles have various special events spaced throughout the year, ranging from parties to special guest speakers. These events break up the routine of weekly presentations, provide a source of focus and anticipation about which to rally, and add momentum to the year, all of which contribute to healthy attendance.

Doubtless an organizer will want to drum up enthusiasm for an upcoming special event, which leads naturally to yet another means of retaining students: staying in regular touch. Several methods of communicating with participants have already been advanced—choose the method that best fits the group, then put it to good use. (But not overuse.) It is helpful for a student who has missed a few sessions to receive an email announcement regarding the need to bring a ruler and compass to the next gathering or reminding them of the special guest lecture for AMC-12 contest preparation. Students inevitably have legitimate reasons for missing several sessions in a row, at which point a potential barrier to renewing regular attendance materializes. The goal is to counteract this hurdle by offering students good reasons to come to at least one more math circle session. High school students in particular are bombarded with many demands on their time, and often need an extra nudge to attend an event that they would sincerely enjoy and benefit from once they arrived.

Keeping in good contact

Speakers may begin to unwittingly lower their level of exposition in response to young faces and elementary questions.

3.2 The Fountain of Youth

Legend has it that Ponce de León discovered Florida while searching for the Fountain of Youth. While this tale is almost certainly not true, the belief in an elixir that will restore youthful vitality or even reverse the aging process continues to persist. And while most math circle coordinators would welcome a rejuvenation of mathematical energy among its participants, not everyone is interested in decreasing their actual age. Yet this is precisely what can and does happen, at least to the average age of the students.

The groups most vulnerable to this Fountain of Youth effect are those aimed at high school students. As noted previously, this age group is attractive to coordinators in a college math department for several reasons. For instance, students are closer in age and mathematical maturity to those whom the coordinator are accustomed to teaching. In addition, the list of potential mathematical topics which could be presented to such a group includes many more fascinating ideas than would be available to a middle school audience. Why then are math circles devoted solely to high school students so scarce?

High school circles look good on paper

The Stanford Math Circle provides a typical illustration of how the Fountain of Youth effect operates. Prior to its first meeting the circle was advertised only to high school students and teachers. Topics selected during the fall (including Gaussian integers, geometry of complex numbers, and topology) were tailored for an older audience. However, word quickly spread among parents looking for mathematical enrichment for their middle school student with a scientific inclination and light homework load. A tally taken of all students who attended the Stanford Math Circle at any point during the winter session revealed that approximately a quarter of them were in eighth grade or below. Several factors help to guarantee their presence: middle school students typically have fewer responsibilities and hence more free time to devote to such activities, they are often bright enough to understand enough of what is presented to be truly captivated by the material, and they are still substantially influenced by their parents' decisions regarding their activities and reliant upon them for transportation. Indeed, some of the most consistent participants that winter were also the youngest.

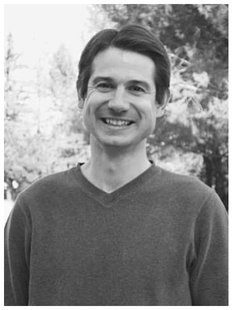
Why do middle school students appear?

Middle school math circle students don't necessarily become high school math circle participants.

Once a significant proportion of the audience falls into the under-fourteen age range, two things are prone to happen. First, high school students may start to feel self-conscious about hanging out with younger kids. Expressing a public interest in math is uncool enough at many schools; doing math with middle school students might be too much. It doesn't help when younger siblings tag along, and the situation becomes even more uncomfortable when there is a little hotshot who is actually as quick at answering questions and coming up with bright ideas as

The effect of a middle school contingent

Circle Snapshots



Name: The Stanford Math Circle

Location: Stanford University, Stanford, CA

Director: Sam Vandervelde

Email: samv@math.stanford.edu

Meeting time: Sundays 2:00–4:00pm

Web site: www.stanfordmathcircle.org

Shortly after completing his graduate studies, Sam Vandervelde found himself in Palo Alto when his wife began a post-doctoral fellowship at Stanford University. Since he had always enjoyed presenting exciting high school mathematics and also had an informal affiliation with Stanford, he jumped at the suggestion made by a friend during the summer of 2005 to begin a math circle. In a month's time supporters sent email out to a large number of local high school math aficionados, Professor Ravi Vakil became the official faculty sponsor and arranged for space in which to meet, and Sam cobbled together a web site, a few forms, and an introductory set of presentations. Over forty students attended the first meeting and the math circle is still going strong, now in its second year.

Between twenty and thirty students converge on Stanford on a typical Sunday afternoon. Sam begins each two hour session with a brief warm-up (a nice problem, puzzle, or idea) before making announcements and introducing the speaker for the day. The rest of the time is devoted to the pursuit of great mathematics, with a short break in the middle for snacks, which are supplied by parent volunteers. The fall ice-cream social and winter pizza party are quickly becoming annual traditions, as are a day devoted to an action packed competition of some sort, an AIME practice session, and a trip to the Bay Area Mathematical Olympiad awards ceremony.

The Stanford Math Circle operates on a yearly budget of approximately $1200, which allows for modest honorariums for speakers, food for the social events, and other incidental expenses. The budget is funded entirely by parental donations, which are managed by the Art of Problem Solving Foundation. Registration data indicates that the majority of students are in grades eight through eleven. An email list is maintained and students are kept abreast of news via occasional email announcements.

the older students. At a more significant level, speakers may also start to unwittingly lower their level of exposition in response to the young faces and elementary questions. Since the younger students are often the most enthusiastic to raise their hands, this can be an issue even for speakers who are presenting more advanced material. Math circle coordinators are especially susceptible to watering down material since they are the most familiar with the group and most eager to boost attendance by ensuring that the mathematics is accessible to everyone.

Responding to younger kids

There are several remedies for this situation which help middle school students feel welcome without discouraging the older students. One of the simplest is to encourage parents or other adults to attend, thus artificially raising the average age. Once the math circle setting feels less like a school classroom and more like an event of interest to the general mathematical public, the issue of age range will become less prominent. It is also important to maintain a steady diet of advanced topics (but not too advanced!), even at the risk of leaving behind some of the middle school students. They are not as easy to intimidate as most coordinators believe, and the ones who are willing to be stretched will benefit a great deal from the presentations regardless. It is good policy to set clear expectations during the first meeting with respect to how much of the material students are likely to absorb. At the Berkeley Math Circle newcomers are advised that to follow even a third of any given talk is a significant achievement.

> *Those on a mission to spark interest in math will want to focus their efforts on middle schoolers.*

Matriculation

Occasionally students who attend for several years may reach the point where they feel they have mastered the math circle curriculum, such as it may be. Thus a Stanford Math Circle parent raised the concern, in a very diplomatic and constructive manner, that material was being pitched too low for his olympiad caliber daughter. The discussion was productive and led to the conclusion that, rather than raising the level of exposition and leaving the majority of the remaining students behind, it was time for this particular girl to move on to more advanced offerings. (These might include top flight summer programs, regional or national olympiads, or college courses, for instance.) In a sense this scenario is ultimately the goal of any math circle: to attract students interested in mathematics and develop them into such proficient and experienced problem-solvers that they are ready for greater challenges.

Divide and conquer

Yet another recourse for math circles with a sufficiently large audience involves splitting the group into two or more tracks based on some combination of age and mathematical experience. This has proven to be a successful approach in Boston, Berkeley, San Diego, and San Francisco, for example. Many aspects of coordinating such a circle, such as filling the speaking schedule, become twice as taxing as before, so be sure to muster additional staffing support. Ideally a secondary coordinator could take over the middle school group, as occurred at the Stanford Math Circle.

Factors that influence attendance

It is worth mentioning that the Fountain of Youth effect doesn't automatically resolve itself over time as students grow older. In other words, middle school math circle participants don't necessarily become high school math circle participants. A math circle might be the only outlet for younger students interested in science; as they grow older more opportunities become available and the same students may opt for the robotics team or become engrossed in developing their school's web site. Similarly, as students become increasingly independent of their parents they may decide that they are actually more interested in the a cappella group or drama club than in math. Yet another factor at work to discourage high school attendance is the fact that participation in a math circle is not perceived to carry nearly as much weight on a college resume as does recognition of achievement in competition, even though these prizes are often much less meaningful. Some math circles have incorporated olympiads or other types of productive contests into their circles, not only to give kids an arena in which to concentrate on solving challenging problems, but also to be able to make awards to their top students. Another organizer is planning to act as a mentor to older students interested in developing math projects for science fairs or papers for talent searches. Students often need direction when developing fruitful problems to tackle. They would then work independently to discover and present solutions.

Why to consider math olympiads

Mission to middle schools

To be fair, there are many excellent reasons to arrange a math circle for younger students, say in grades five through eight. This window of time is arguably the most effective one for spreading the mathematical gospel, since many students make up their minds in terms of whether they "like math" or are "good at math" during these years. In fact, individuals on a mission to spark interest in the subject will probably want to focus their efforts on a middle school audience. The point is that coordinators interested in reaching older kids must be proactive in finding ways to keep a math circle targeted for high school students attractive to that age group.

3.3 MyMathCircle

The discussion so far has centered primarily about the fundamentals of math circle operation, such as scheduling, publicity, and funding. But now we interrupt the relentless stream of logistics to consider some of the less essential facets of a math circle. There are any number of ways that coordinators can add their own personal touch, ranging from a clever logo to a lending library. These extras contribute far more to a circle than the length of this section might suggest. They are likely to represent the coordinator's favorite aspects of the undertaking; these are the elements of the math circle which make organizing it a delight rather than a burden. Below is a cross-section of ideas taken from various math circles.

Personalizing a math circle

- The Berkeley Math Circle holds a monthly "Winner's Handicap" contest, run by a graduate student, in which participants are given four weeks to compose solutions to a set of five problems. The coordinators evaluate all the papers and award prizes to the top students. Then in subsequent months the stronger students are handicapped based on their scores on previous contests to help level the playing field and encourage the less experienced students to continue to work diligently on the problems. The Berkeley Math Circle has also created a nice logo involving a verdant tree featuring the digits of pi superimposed across its leaves with the Golden Gate Bridge in the background. This graphic appears at the top of their web site and on other materials associated with the math circle.

- The St. Mark's Institute of Math puts out a monthly newsletter featuring an introductory mathematical morsel, an interesting exposition, and a variety of questions (even open problems!), all centered about a common theme. The newsletter is published in hardcopy form only; the purpose is to populate the coffee tables in teachers' lounges across the greater Boston area with these brightly colored sheets of paper in the hopes that they will engage many more people than appear on the mailing list. The newsletter is currently mailed out to approximately 600 teachers and students.

- The Washington University Math Circle likes to send all participants home each week with an item related to the math that has just transpired. This might be a worksheet of problems for further investigation, or it could be some other type of souvenir, such as a set of linked Mbius strips resulting from an investigation in topology.

- The Stanford Math Circle presents warm-ups during the first ten minutes of each meeting. These might range from digit puzzles ("How many ways are there to make a perfect square with exactly four 1's?") to more meaty mathematical morsels ("How is it possible to fill all of three-dimensional space with non-overlapping circles?") In this manner the meeting can begin promptly while still allowing leeway for latecomers to hear the entire presentation.

- The San Jose Math Circle wheels in a large collection of math texts and problem books each week written for secondary students. Students can browse titles during any free moments and then check out books on the honor system. The coordinator also regularly distributes other items of interest, such as past issues of *Math Horizons*, the MAA journal produced for undergraduates.

- The Mobile Math Circle holds an internal olympiad in early March as a culmination of the many problem solving sessions they conduct throughout the year. The organizers recognize all the students who made significant progress, presents book awards to a couple of high scoring participants, and rewards their top two students with an all expenses paid trip to the Colorado Math Olympiad event later that spring.

- The San Francisco Math Circle hosts a year-end picnic. Families of all the participating students are invited to gather at Golden Gate Park to spend an afternoon together. Besides engaging in the expected outdoor activities, kids also take part in an informal ceremony recognizing outstanding participants.

- The Sudbury Math Circle, which is based at a private school in Sudbury, Ontario, devotes an entire school day to holding their annual "Math Challenge Day" for their own students and any others who are able to attend. The students take a math contest in the morning, followed by a math circle event from an invited speaker. A local pizza sponsor provides lunch, after which the students have recess and hear a second presentation by the speaker. Parents arrive, prizes are awarded, a reception is consumed, and everyone returns home exhausted but elated.

- The San Diego Math Circle runs math games as part of each meeting. Students attend a presentation during one half of the Saturday meeting and take part in an exciting, high energy team event during the other half. There is a snack break separating the two halves of the morning. This format permits speakers to work with each age group. The San Diego kids also sport cool T-shirts.

3.4 Paperwork

This section will be kept short and sweet. Administrative tasks may not top every coordinator's list of favorite activities, but there is no substitute for being prepared ahead of time with useful forms to ensure that a math circle runs smoothly. Some common aspects of a circle's operation which may require preparing a form are listed below. Samples of many of these appear in the corresponding appendix to provide a guide for preparation. The only one that is really essential for a moderately sized, informal math circle is the registration form; include others as necessary.

It is good practice to get in touch with speakers to fill them in on what to expect on the big day.

Be prepared

Organizers might choose to group information by student, so that each person attending receives a single sheet of paper which contains details about the math circle, lists ways in which families can get involved, provides instructions for donations or tuition, requests information about the participant, and perhaps includes waivers and space for signatures. Alternatively, the organizer could collect information by topic, so that one sheet of paper contains all registration information, another is used for attendance, a third handles volunteer sign-up, and so on. Each approach has its advantages; for example, the latter option makes more sense if a helper will be creating an email list, since all the necessary information will already be together in one place. Choose whichever option seems most appropriate.

Options for disseminating information

- ☐ **Registration.** It is extremely useful to track information on all individuals who attend the math circle. At the very least it is helpful to know each student's name, grade level, school name, and email address. Declare unequivocally at the top of the form that students' personal contact information will not be distributed to any other individuals or organizations, then abide by this policy. In the event that someone does have a legitimate reason to reach a student, offer to contact that student on their behalf so that the student has a choice of whether or not to respond.

- ☐ **Attendance.** Recording who is present serves many purposes. Knowing total attendance each week and the change of this quantity over time provides a rough gauge of student interest and can be important to include in a report of the circle to those funding it. It is also helpful to know the number of students at each grade level who are participating on a regular basis. In addition, one can present awards for perfect (or nearly so) attendance.

- ☐ **Welcome.** In lieu of meeting with parents, an organizer might prepare a letter for them to pick up the first time they attend. This letter could begin by expressing appreciation for transporting kids and describing the wonderful speakers lined up for the year. It could then go on to provide an overview of the circle's operation, discuss ways that parents could volunteer or otherwise become involved with the circle, and explain how donations or tuition works, if applicable.

- ☐ **Volunteering.** One means of obtaining volunteer support for chores such as providing snacks each week is to set out a form asking parents to sign up for various dates.

Leave space for volunteers to provide their name, phone, and email address for each date or task. Another creative approach (inspired by a kindergarten teacher) consists of writing "wish list" items on post-it notes so that parents can sign up for providing items or performing duties for the circle by simply removing a note and taking it with them. These requests might include one prize for the monthly contest, signs for directing new students to the room, email list maintenance, or anything else that a coordinator needs on a regular basis.

- ☐ **Consent forms.** Some math circles will need students to sign waivers in order to abide by university policies for use of rooms or for insurance purposes. These may include agreements not to sue the university in case of injury or to abide by all math circle policies. It is also necessary to obtain consent from parents of minors to post their pictures at the web site or include them in video clips.

Weekly tasks

Besides keeping attendance and updating email lists, there are a few other weekly administrative tasks that bear brief mention. For example, it is good practice to get in touch with upcoming speakers to remind them of their presentations, arrange for any special needs such as a laptop projector, request a sheet of additional problems, and generally fill them in on what to expect on the big day. In the same manner, some organizers like to send a description of upcoming meetings and make announcements each week to all students on the email list. Other weekly tasks might involve updating the web site with the latest handouts or assembling questions for warm-ups or math games. No doubt each organizer will quickly develop their own weekly routine.

3.5 How am I doing?

Why to find out

Obtaining feedback on a math circle is a fairly natural impulse for a coordinator. This person has initiated the circle with a particular purpose in mind, presumably out of a desire to provide an exciting mathematical opportunity for secondary students. So it stands to reason that the coordinator will be interested in finding out how successfully this purpose is being achieved. Although the director probably has a fair grasp of the overall effectiveness of the circle, it is very easy for this person to get caught up in organizational details and miss important trends or shortcomings. Therefore it is important to pause periodically to solicit friendly but frank feedback, or otherwise objectively gauge the health of the math circle.

Praises and gripes are freshest in students' minds immediately following the presentation, so they are most likely to express their feelings to their friends outside the building or to their parents during the trip home. Hence parents can be an excellent source of information. Make a deliberate effort to say hello occasionally to parents standing alone or (better yet) chatting with other parents. Inquire whether their kids are enjoying the math circle and ask directly if the parents would recommend any changes based on what they hear from their children. This form of question avoids the dilemma for parents of revealing students' criticisms verbatim, but keeps the focus on the kids as opposed to opening the floor to suggestions for steering the circle towards the parents' vision of what a math circle ought to provide for their college-bound son or daughter.

Don't wait until the end of the year to take stock; the time is ripe five or six meetings into the schedule.

Getting inside information

The other method of discovering students' reaction to the math circle is to ask them by way of an anonymous survey. Limit the questions to those items that are most important. Most students only have the patience to answer one or two open-ended questions thoughtfully, so make them count. The survey might cover aspects of the circle such as the level of the material (too easy or hard?), the topics covered (routine or fascinating?), the style of presentation (droning or engaging?), what the students liked, and what they would change. For example, after conducting a survey the coordinators of the Washington University Math Circle discovered that students were regularly lost during talks, and were able to adjust the level accordingly. Don't wait until the end of the year to take stock; the time is ripe at any point five or six meetings into the schedule. As motivation, make a completed survey the students' ticket for snack or pizza that day. A sample form is shown in the appendices.

Scintillating surveys

Coordinators who will be making reports for individual or corporate donors supporting the circle need to be even more deliberate about gathering feedback. In addition to the approaches described above, it is important to settle on an objective yardstick for measuring the success of the circle before the academic year gets underway. Attendance figures and breakdown of audience composition by gender and race could all be important, depending on the stated goals of the original funding proposal. Be sure to write down positive student and parental testimonials as soon as possible after hearing them; they fade from memory all too quickly. Take pictures of students actively engaged in mathematics. Record any honors garnered by students at the math circle as they come to light; this item might also be included on a follow-up year end survey. The sample grant report in the appendices can also suggest further data to collect.

Planning for grant reports

3.6 Paging all parents

Several math circles, including those in San Diego and Berkeley, host a yearly meeting especially for parents of students attending the circle. This event is usually held at the same time as the regularly scheduled math circle at some point during the autumn; perhaps during October or early November. Such a meeting is a sound idea for any math circle which relies on parents in a significant way, financially or otherwise. This sort of gathering is particularly well-suited for circles in which regular contact with parents is minimal or non-existent; for example, if parents are only able to drop students off at the door of the building due to parking restrictions.

Why to schedule parent meetings

There is no substitute for testimonials when it comes to securing support.

There are many good reasons to make an annual tradition out of meeting with the adults associated with a math circle. Just as with Back-to-School nights, there will probably be no better chance to describe in glowing terms the positive mathematical impact a math circle can have on students and to outline exactly how parent's contributions are translated into bringing about this experience. In the same breath, one could call for volunteers to help with the math circle. Finally, parents would appreciate the chance to ask questions and learn first-hand what their children have been doing all those weeks.

In light of these purposes, a sample agenda for the fall parent meeting might proceed as follows. For starters, acknowledge all

Circle Snapshots

Name: The Berkeley Math Circle

Location: University of California, Berkeley

Director: Zvezdelina Stankova

Email: `stankova@math.berkeley.edu`

Meeting time: Tuesdays 6:00–8:00pm

Web site: `mathcircle.berkeley.edu`

Zvezdelina Stankova is a Bulgarian math circle success story. As a fifth grader she was originally drawn to her middle school math circle when she discovered that her peers were learning how to approach tough problems that she was not able to complete. She rapidly became very adept at problem solving and ultimately represented Bulgaria at the 1987 and 1988 International Mathematical Olympiads held in Cuba and Australia.

College and graduate school brought Zvezda to the United States, but her involvement with math circles started only after her arrival in Berkeley, where she began her post-doctoral work at MSRI in 1997. During her first year she helped to advertise the concept of math circles to secondary school teachers in the area. When it became clear that most teachers did not have the requisite background, she decided to coordinate a temporary math circle at the University of California, Berkeley which would serve as a model for teachers. This "temporary" circle is now entering its ninth year.

Several years after commencing it became clear that a select group of highly talented and motivated kids would benefit from more advanced topics and a greater emphasis on problem solving. So for a while two groups met concurrently. One of the highlights of being the coordinator occurred when two of her participants represented the United States at the IMO; in a sense Zvezda had come full circle. Not long after the demands of offering an advanced track became too great and the group transitioned to its current state in which it is comprised mainly of seventh through eleventh graders.

The Berkeley Math Circle began with a $1000 budget to pay for prizes for their monthly math contest and cover administrative costs. It then ran purely on volunteer enthusiasm for five years before MSRI located outside donors to help expand the scope of the program. As a result Zvezda is now co-director of the Bay Area Math Olympiad in addition to being the director of the Berkeley Math Circle.

the people and organizations who have made the math circle possible. This would be a good time to invite a representative to say a few words on behalf of a major donor, if applicable. One might also recognize the host institution, regular instructors, and volunteers who have helped out in the past. The next logical step is to give an idea of what transpires in a typical math circle. The most effective means would be to conduct a miniature ten-minute presentation on a suitably enticing yet elementary topic. Alternatively (or additionally) a student or two could talk about what usually takes place and why they enjoy participating. There is no substitute for testimonials when it comes to securing support.

A sample parent meeting agenda

When the San Diego Math Circle enlisted the aid of its parents, the size of the group exploded from less than twenty to around one hundred fifty!

Once parents have a sense of what occurs and why it is worthwhile, take a few minutes to explain in clear and specific terms how parents can support the math circle. These might include details on how to make donations, where to sign up for bringing snacks, what transportation needs exist, or whether parents could help to advertise the circle at local schools to build the program. Finally, open the floor for questions and comments. Ask parents if they have any observations or suggestions to make based on feedback they have obtained from their children. Aside from fielding questions, the end of the meeting is also an appropriate time for alerting parents to other resources for their children, such as math olympiads or summer programs that the organizers endorse.

Following up

When rallied effectively, parents can be a powerful force for advancing a math circle. When the San Diego Math Circle enlisted the aid of their parents in order to increase attendance, the size of the group exploded from less than twenty to around one hundred fifty students! Berkeley Math Circle parents organized a subsequent meeting of their own initiative in which they hatched a plan for a Christmas party where students would create mathematically-themed tree ornaments and enjoy cookies and punch. The moral of this story is not to underestimate the resources available to math circle organizers in the form of parental support, and to seriously consider holding a meeting to mobilize this support each fall.

The power of parent support

3.7 Going the distance

At an opportune moment, pause to reflect on what aspect of coordinating a math circle is most appealing. Whether this be a love of teaching, a desire to reach underserved kids, or an interest in bringing students to campus, it is important for coordinators to identify the elements of the math circle that they find to be the most motivating and inspiring and then ensure that these elements remain at the center of their involvement with the math circle. Circumstances inevitably change, and it is surprising how easily coordinators can find themselves spending the majority of their time dealing with tasks (typically administrative in nature) which are draining rather than energizing. When this situation develops it is crucial to recall the original source of inspiration for taking charge of the math circle and find ways to regain this role.

Keeping in mind one's original motivation

> *Math circles endure when directors continue to find the challenge of organizing them year after year to be stimulating and rewarding.*

For example, a person who enjoys presenting sophisticated topics to a tight knit, internally motivated group of students will face a challenge when the circle becomes more popular and attracts the attention of parents who bring kids for the sake of improving their math skills. Rather than becoming resigned to watering down material for this new, wider audience, the coordinator must dream up creative means for pursuing their original vision while simultaneously being diplomatic. One possible solution would be to arrange for a "fun group" and a "serious group," although not in so many words. (They could be dubbed the "Pythagoras" and "Archimedes" groups, for example.) The former group could make soap bubbles with Zome tools, learn basic counting techniques, and meet for a shorter length of time. The latter group, on the other hand, would be expected to complete a challenging weekly problem set, actively take part in a discussion of these problems, and tackle topics such as inversion or generating functions. To avoid doubling administrative duties, oversight of the fun group could be completely delegated to the parents.

First dilemma

Another coordinator might take pride in the fifty minutes worth of math games that are run each meeting for the kids. Although this sort of activity requires some advance preparation in terms of assembling and photocopying questions, it is easily justified by the level of enthusiasm and active participation it produces in the students. This person would be dismayed, therefore, if the size of the group doubled or tripled so that preparing for the math games became an unwelcome weekly burden. Rather than discontinuing the games or despairing at what has now become a chore, this coordinator might look for solutions such as raising funds via the parents to pay a team of graduate students to run the games. Another alternative might be to arrange for the more advanced students to host the games every other week for the rest of the group.

Second dilemma

As a final example, consider a coordinator who initiates a circle at his university to serve kids who are interested in math but who have not had much exposure to extra-curricular topics. This person would be understandably distressed if the math circle caught on among nearby private schools so that kids with a much broader mathematical background began to dominate the discussions and scare off the target audience. One practical solution would be to change location in such a way that the event simply becomes much more accessible to the target audience. If feasible, the original circle could continue for all interested students under the leadership of another person, while the founding coordinator could establish a second circle at the new site.

Third dilemma

All of these illustrations are meant to suggest that a little healthy self-interest on the part of the coordinator is good for the life of the circle and will help to ensure that it can weather the inevitable changes in circumstances that will arise. There are certainly other factors which also contribute to the longevity of a math circle, such as a tight-knit leadership team, supportive parents, or stable funding. But a math circle is often initiated through the vision of one or two people, and to persist (at least in its original incarnation) it is important that these individuals are able to keep sight of their original motivation as the math circle develops over the years. In other words, math circles endure when directors continue to find the challenge of organizing them year after year to be stimulating and rewarding.

Enduring math circles

Chapter 6

Double Time

Binary representation ($\star$)

Overview. At one level, binary is just arithmetic performed with only the two digits 0 and 1, as opposed to the customary ten digits 0, 1, 2, . . . , 9. Since students are already familiar with performing calculations in our standard base ten notation, they can enjoy discovering the process for adding and multiplying binary numbers without requiring much direct instruction. As students will find, binary arithmetic is based on powers of two, so binary is directly or indirectly related to almost any process these powers. Examples abound and include efficient systems of weights, programming algorithms, and many popular mathematical puzzles, such as the Towers of Hanoi. I like to introduce this concept to math circle students through an exercise which helps them to really get the feel of binary.

Activity. Any group of four people can take part in this activity, called "Last One Standing." The four individuals should sit in a row of four adjacent seats, all facing the same direction. If possible, they should orient themselves as if they were standing in a line, so that the person in the rear can see the other three, while the person in front can't see anyone else. It is also fine to sit shoulder to shoulder, all facing outward, as if seated on a couch. Just replace 'in front' and 'behind' with 'to the left' and 'to the right' in these directions.

To begin, all individuals should be seated. The object is to arrange for the last person in line to be standing, while all others are seated. The participants can move according to a single rule: a person can change state (by standing up or sitting down) if the person immediately in front of them is standing, while all others in front of them are seated. Otherwise they are locked in position and may not move. The status of people behind them does not matter. The person in front, to whom this rule does not apply, can stand or sit at will.

Problems. During the math circle students will consider some of the following questions pertaining to the "Last One Standing" activity. They can then move on to other questions related to binary numbers and arithmetic.

1. How many steps are needed to accomplish the task?

2. If we allow more than four people to sit in a row, then what is the fewest number of moves needed to have the last person standing and all others seated?

3. How many times does a particular person in the row have to move in the course of an optimal solution?

4. How long would it take a group of twenty people to complete the activity?

5. Based on the pattern, determine the next ten "numbers" in the sequence

$$1, 10, 11, 100, 101, 110, \ldots$$

6. What is the 75^{th} number in this sequence? What about the 99^{th} number?

7. Where in the sequence does the number 101010 appear? How about 1010001?

8. Find the 75^{th} number in this sequence by adding 11001 (the 25^{th} number) and 110010 (the 50^{th} number).

9. Find the 99^{th} number in this sequence by multiplying 1001 (the 9^{th} number) and 1011 (the 11^{th} number).

Presentation Notes. Introduce the "Last One Standing" activity by having four volunteers sit in a row at the front of the room or some other conspicuous location. Explain the rules of the activity, then allow the group to attempt the task. Encourage the rest of the students to maintain a respectful silence as long as the volunteers are making progress. However, be sure to immediately interrupt their efforts if an illegal move is made, then ask the audience where they went astray and what move should have been made instead. Applaud your brave volunteers when they finally successfully accomplish the activity.

It is rather important to rehearse this activity yourself ahead of time. For example, you can use the index and middle fingers of each hand to simulate the four individuals. The first person will stand, which allows the second person to stand. Next the first person sits back down, so that the third person can stand. Then the first person stands again, permitting the second person to sit, and so on. The entire activity will require fifteen steps and have a distinctly binary feel to it.

Then divide all the students into groups of four and have them practice on their own until they are able to finish the task smoothly. (Leftover students can be assigned to existing groups to act as coaches or periodically rotate into the lineup.) When a group is ready to demonstrate the steps, surprise them by pulling out a stopwatch and timing their solution. I would recommend giving each group at most two attempts to get the last person standing; if they make an invalid move they have to start over from the beginning. Keep track of the best times at the front of the room until just before students begin to tire of the activity. Then award a prize to the fastest group and return the room to its former state.

Next instruct the participants to ask mathematical questions about the activity that just took place. This is one of the most important parts of the math circle, since it gives students the opportunity to engage in a fundamental aspect of any mathematical investigation; namely, asking fruitful questions. You will probably need to be patient, since students don't have much experience with this process. If needed, direct them to think about what quantities related to the activity could be counted. Ask them if they can generalize their questions. Given time, they should be able to pose questions similar to the first four stated above. I include "How long would it take a group of twenty people to complete the activity?" in their list because it gives a good focus to the discussion and has an unexpected answer.

Students can proceed to find answers to the questions they just posed. Emphasize that one of the best ways to solve hard problems is to formulate and answer easier versions of the question or examine simple special cases. For example, this strategy could be employed for finding the number of steps required for a group of twenty to complete the activity. A table which summarizes this data might look like the one below.

group size	1	2	3	4	5	6
steps needed	1	3	7	15	?	?

Students can then find patterns to predict the next couple of entries in the table. Responses usually include "Double each number and add 1 to obtain the next number," or "The difference between successive entries is always a power of 2," along with less helpful ideas. Someone might also notice that the entries in the bottom row are all 1 less than a power of 2. (If not, steer them to this observation or make it yourself.) Compare and contrast these ways of looking at the data, and discuss the advantages and disadvantages of each. Thus the first idea provides a recursive way of generating entries and provides a natural way of understanding the pattern. On the other hand, the last idea gives an explicit formula and is more useful for computing the twentieth entry. Stress that finding patterns is only the beginning of the story; the real mathematics lies in proving that our patterns persist. While noticing the patterns will allow us to compute answers, the underlying mathematics contains the beautiful ideas that make the whole enterprise worthwhile.

By this time students will have conjectured that it will take five people 31 moves to get the last person standing alone. Have five volunteers give this a try and count the steps to confirm this prediction. Then try again, this time allowing them to skip straight to the step in which the fourth person is standing, since they are well-acquainted with how to accomplish this in 15 steps. The fifth person can then stand on the sixteenth move. Ask how many more steps it will take to result in the fourth person sitting back down. Once they realize that it will take 15 more steps (same as the initial 15 steps, except the fourth person sits instead of stands at the appropriate moment) then they have essentially grasped the recursive nature of the process. To cement the idea, add a sixth person at the end of the line and ask how many steps are required in this scenario. The answer is 63: 31 steps to have the fifth person standing alone, 1 step for the sixth person to stand, then 31 more steps to seat the fifth person again. This explains the "double and add 1" observation.

At your discretion, use the recursive formula to prove the explicit formula. In other words, suppose we have a number that is 1 less than a power of 2. We wish to explain why doubling this number and adding 1 results in another number that is also 1 less than a power of 2. I usually begin with

$$15 = 16-1 \quad \text{so} \quad 2(15)+1 = 2(16-1)+1 = 32-2+1 = 32-1 = 31,$$

then introduce variables to polish off the general case:

$$2(2^n - 1) + 1 = 2^{n+1} - 2 + 1 = 2^{n+1} - 1.$$

Regardless, at least mention that the recursive formula can be used to derive the explicit formula. At this point the group will be able to calculate how long it will take a group of twenty to finish the task. (It is entertaining to have them guess first. Most estimates are less than a couple of hours, which is well below the actual duration.) Use the winning time from above to estimate the time needed per move; a half second is a reasonable amount. Using this figure, it would take a little over six solid days to finish, with no breaks for eating or sleeping, at least for the first person in line!

During the remainder of the math circle let the students develop as much binary arithmetic as they are able. The hardest part about leading this activity is to resist the temptation to present it all yourself. One possible sequence of questions is outlined in the problem set above; this has proven to be an effective approach in other math circles. Here are a few ideas to help guide the discussion.

- Discourage students from referring to the binary numbers 10, 11, and 100 as 'ten,' 'eleven,' and 'one hundred,' but call them 'one-zero,' 'one-one,' and 'one-zero-zero' instead.

- Make sure they notice and appreciate the fact that the "special" numbers 10, 100, 1000, and 10000 occur in the second, fourth, eighth, and sixteenth positions in the sequence.

- Based on the previous item, ask for a term of the sequence near the 75$^{\text{th}}$ term that they know for sure, then work from there. Help the group appreciate that $75 = 64 + 11$ translates into $1001011 = 1000000 + 1011$.

- In general, finding the powers of 2 used to "build" a number gives a shortcut to finding a term in the sequence. For instance, since $99 = 64+32+2+1$ we deduce that the 99$^{\text{th}}$ term will be 1100011.

- Try this algorithm in reverse in the next question to find that 101010 is the 42nd term and 1010001 is the 81st term.
- Gradually transition from treating the strings of 0's and 1's as terms in a sequence to thinking of them as legitimate numbers. Ask what sorts of operations one can perform with numbers (addition, multiplication, etc.) and suggest that we try these operations on binary numbers as well.
- Based on their knowledge of arithmetic, see if the students can figure out a few simple examples such as $101 + 1 = 110$ and $1011 + 111 = 10010$ on their own. Review the process of carrying together, then let them try the next question in the problem set above.
- Similarly, see if they can compute $1011 \cdot 101 = 110111$ before illustrating longhand multiplication for everyone. They can then check their earlier prediction for the 99th term by answering the final question.

Other avenues of exploration could be based on the problems below, or you could insert your favorite binary activities at this point. For instance, one could discuss how and why computers utilize binary, continue with the Towers of Hanoi puzzle, investigate Gray code, or just wrap up. I find that students are consistently more fascinated by binary than I expected. So have a great time with this topic in your math circle.

Further Problems.

1. List the next five binary numbers after 1001011.

2. Convert the unlucky numbers 13 and 666 into binary.

3. Compute the sum $1110010100101 + 1011011011101$ in binary, writing your answer in binary. (Don't even think about converting them to decimal numbers.)

4. Find a pair of binary numbers, each with exactly three 1's in their binary representations, such that their sum has only a single 1 among its digits.

5. Perform the multiplication $10001001 \cdot 1101001$ in binary.

6. Find a four-digit binary number such that placing an extra 1 at the beginning and end results in a new six-digit number which is five times as large.

7. Merchants used to weigh items using a set of standard weights and a balance scale. For example, with nine 1-oz weights and nine 10-oz weights a vendor could measure any amount from 1 to 99 ounces by placing the item on one side of the scale and certain weights on the other side. However, this requires hauling around eighteen weights. Can you figure out how to accomplish the same thing with only seven weights? (In this problem assume that the weights may only be placed on one side of the scale, while the item goes on the other side.)

8. If you found the solution that I'm thinking you did to the previous problem, then your weights total 127 ounces—significantly more than the 99 ounces that our merchant had to carry before. Now figure out how to weigh any amount from 1 to 99 ounces with seven weights having the smallest possible total weight.

9. If we include leading zeroes, then the binary numbers from 0 to 7 look like

$$000, 001, 010, 011, 100, 101, 110, \text{ and } 111.$$

Arrange these eight numbers in a circle so that every pair of adjacent numbers has identical digits except in one spot. (For example, 110 could be next to 010 but not 011.)

10. Once you have solved the above problem, can you think of a shortcut for achieving the same thing with the sixteen binary numbers from 0000 to 1111? (Try to build on the previous solution rather than starting from scratch.)

11. How many ways are there arrange the binary numbers from 0 to 7 in a circle so that adjacent numbers differ in only one digit? Hint: there is a beautiful way to picture this problem using a cube whose vertices have coordinates $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, ..., and $(1,1,1)$.

Hints and Answers.

1. The numbers are 1001100, 1001101, 1001110, 1001111, and 1010000.

2. Written in binary we have 1101 and 1010011010. Not nearly as unlucky-looking as before!

3. The sum comes to 11001110000010.

4. The numbers 10101 and 1011 sum to 100000, for example.

5. The product is 11100000110001.

6. Let N be the sought after binary number. Appending 1's to the front and end of N yields the number $32 + 2N + 1$, since N has four digits in binary. This quantity should be five times as large as N.

7. The solution that naturally presents itself is to use weights of 1, 2, 4, 8, 16, 32, and 64 ounces.

8. The total weight has to be at least 99 ounces in order to weight an item which is that heavy. There are many ways to find seven weights which total exactly 99 ounces. The most obvious, perhaps, is to simply to replace the 64-oz weight in the previous solution with a 36-oz weight.

9. There are many possible answers. One is

$$000, 001, 011, 010, 110, 111, 101, 100.$$

10. Place a 0 in front of all the above numbers for the first eight entries around the circle. Then place a 1 in front of the same numbers and list them next, in reverse order. In other words,

$$0000, 0001, \ldots 0100, 1100, 1101, \ldots, 1000.$$

11. The vertices of the cube represent the binary numbers from 0 to 7 via their coordinates. (For example, the vertex at $(1, 0, 1)$ stands for 5.) The key is to realize that the edges of the cube connect exactly those pairs of vertices which differ in one coordinate/digit. (Draw a picture to see this.) So the question boils down to finding the number of ways to traverse the edges of a cube so that you visit each vertex exactly once and return to your starting point. There are essentially six different ways this can be done; double this if you distinguish the direction in which the path is traced out, and eight times this if you keep track of where the starting point is located.