

Chapter 3

Fibonacci Numbers

We asked our students to guess the next number in the following sequence:



1, 1, 2, 3, 5, 8, ...

One child suggested that the next number is 11 since the resulting sequence represents the ages of the siblings in her family.

Soon, the students found the actual rule: every number after the second one is the sum of the previous two numbers. This sequence is called the *Fibonacci sequence*.

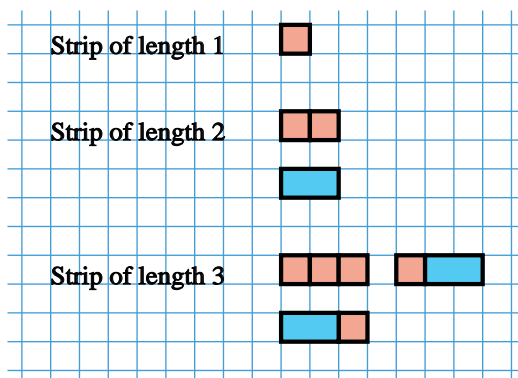
Teacher → To prepare for this chapter we had asked the students to guess the next number in a few sequences a couple of weeks earlier (see warmup Problem 6.10).

Building Strips with Squares and Dominoes

Problem 3.1. Squares and Dominoes. We have two kinds of building blocks: squares  and dominoes . Find the number of ways to build strips of length 1, 2, 3, 4, 5, and 6 out of these blocks.

Teacher → Using colors helps the children to see the patterns and come up with the recursive relation faster.

With the children's help we got the following picture on the board for strips of length 1, 2, and 3:



There is only one way to build a strip of length 1, two ways to build a strip of length 2, and three ways to build a strip of length 3.

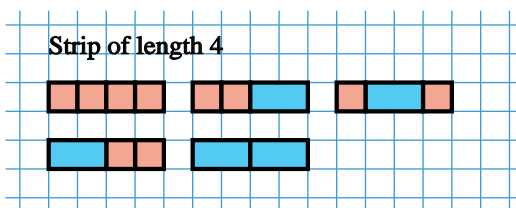
We put these results into a table and asked, “How many ways are there to build a strip of length 4?”

Number of squares in a strip	1	2	3	4	5	6	7	8
Number of ways to build a strip	1	2	3					

Most students immediately yelled four, continuing the obvious numerical pattern. However, this guess turned out to be wrong, since in less than a minute some children found five different ways.

Teacher → For the children this was a striking example of an obvious pattern that should have worked, but didn't. The breaking of such an obvious pattern justified the question: “Will this pattern continue forever?” which appears again and again in our discussions of patterns.

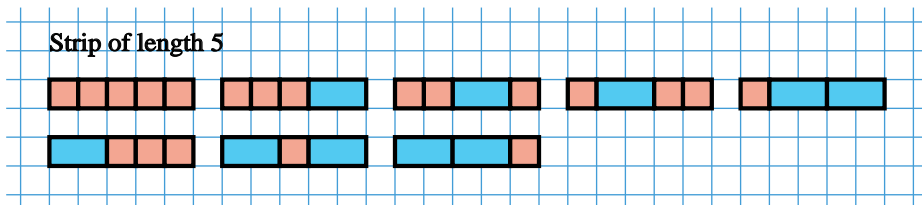
Soon the rest of the children also found five ways to build the strip of length 4, and we summarized the result on the board:



The strip of length 5 took more time. To ensure that all possible ways of building the strip are found and none are repeated, we reminded the students of how important it is to have a systematic way of listing the possibilities (problem “Mumbo 5-Letter Words”, page 26). There are many ways to accomplish this; one of them helps to discover the recurrence relation. We led the class towards it: “The picture on the board follows a certain system. Can you figure this system out?”

Many students observed that in the top row the first building block is always a square, while in the bottom row it is always a domino. The same principle should work inside the strips: always place a square first; then, only if all arrangements starting with a square are already listed, use a domino. Some of the children still had difficulties drawing all possible arrangements, so we suggested denoting squares and dominoes with letters A and B. It allowed the children to follow a familiar dictionary order. Some of them even mentioned that the strips are arranged like the words in a dictionary (problem “Mumbo 5-Letter Words”, page 27).

At last, everybody was able to produce the following pictures for the strip of length 5:



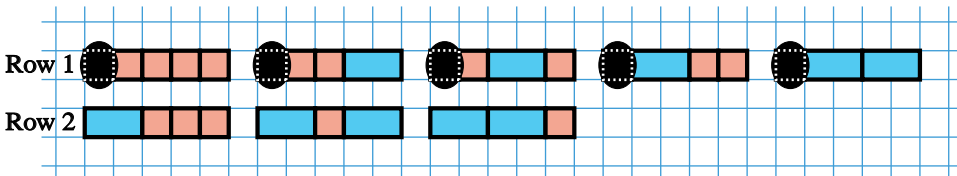
We added the latest results to the table:

Number of squares in a strip	1	2	3	4	5	6	7	8
Number of ways to build a strip	1	2	3	5	8			

At this point our students recognized the Fibonacci sequence discussed at the beginning of this chapter. The prediction for the number of ways to build a strip of length 6 was unanimous: $13 = 5 + 8$. We all agreed that checking this prediction by drawing so many arrangements would take a significant amount of time and is error prone. It might be easier to justify why the next number should always be a sum of the two previous numbers.

Teacher → In many classes the children do not need such detailed hints, as described below. Based on our experience the following is the worst case scenario.

To explain this rule, let us consider the case we are already familiar with, the strip of length 5. Most of our students needed a hint: “What can be the first step in building the strip?” The children suggested to put a square first (for them, first means on the left). We asked, “How should we build the remaining part of the strip?” and covered the leftmost square in every strip in the top row:

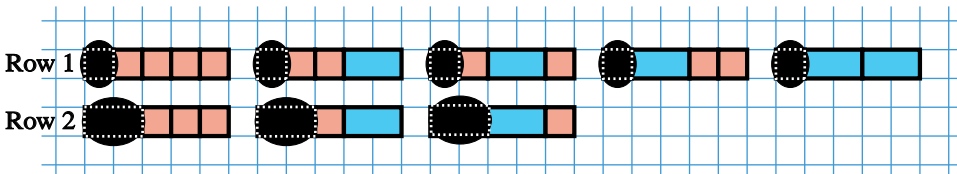


Now a few kids were able to proceed on their own. They recognized that the colors on the uncovered parts of the strips are the same as on the strips of length 4. Several kids still needed additional clues: “What is the length of the remaining part?”; “Do we know how to build the strip of length 4?”

The class concluded that if the first building block is a square, there are five ways to build it. This number is the same as the total number of ways to build the strip of length 4.

We asked the class, “Are we done yet?” Many kids exclaimed, “No, we should find what happens when the first building block is a domino, look

at the bottom row!” They proposed to cover the leftmost domino in every strip of the bottom row:



The kids discovered that the uncovered parts of the strips in the bottom row match three known strips of length 3. They got very excited and yelled, “This means that the pictures of strips of length 3 and 4 ‘are hidden’ in the picture of strips of length 5!” Hence, there are $5 + 3 = 8$ ways to build the strip of length 5.

Finally, the students were able to come up with the following argument for the strip of length 6:

- (1) If the left block of this strip is a square, the remaining part is the strip of length 5. We already know that there are eight ways to build the strip of length 5 with squares and dominoes.
- (2) If the left block of this strip is a domino, the remaining part is the strip of length 4. We already know that there are five ways to build the strip of length 4 with squares and dominoes.
- (3) We have no other choices for the leftmost block! The total number of ways to build the strip of length 6 is the sum: $5 + 8 = 13$.

One can apply the same approach to strips of length 7, 8, 9, This means that every number in the table above is the sum of the two previous numbers. It is the Fibonacci sequence! ■

Teacher → Our students cannot formulate statements which involve unknown quantities. So we did not ask them to state “a general argument”. It is enough if they can see that “the same argument also works for longer strips.”

This was a good opportunity to tell the class that what we did is very similar to the work done by real mathematicians.

- We studied the simplest examples first.
- We figured out the pattern.
- We invented a systematic way of reasoning.
- We came up with an argument that works for all cases: those we considered before, and all the others!

The children were very excited to hear this.

Parking Problems

Teacher → Most children find the next two problems more challenging, although their solutions are the same as for the previous problem.

Problem 3.2. Parking Cars and Motorcycles. A parking lot has 10 narrow parking spaces in a row. A motorcycle takes one space, and a car takes two spaces. How many ways to park cars and motorcycles are there so that all the spaces are taken? What if there are 12 parking spaces in the parking lot? Note that all the cars look exactly the same and all the motorcycles look exactly the same too.

The class started drawing different parking arrangements but soon discovered that there are too many of them. Hint: “Solve a similar but simpler problem first. It will give you a clue for solving the more complex one.” After drawing parking arrangements for lots of length 1, 2, 3, and 4, the children recognized the pictures, “This problem is the same as ‘Squares and Dominoes’.” The answers should be 89 and 233 — just continue the table from the previous problem. However, when we asked the students to explain why the problems are the same, many of them struggled. The following analogy proved helpful.

There are two languages, Blockian and Parkian. Blockian has only two words: “square” and “domino”. Comparing the last two problems, can you translate these words into Parkian?

BLOCKIAN	PARKIAN
Square	?
Domino	?

In a few minutes the students figured out that a “square” should be a “motorcycle” and a “domino” should be a “car”. This translation establishes a way to turn the strips of squares and dominoes into ways of parking motorcycles and cars. We can translate in both directions, which means the number of ways to build a strip must be the same as the number of ways to park. As we found earlier the Fibonacci sequence is the answer to Problem 3.1, “Squares and Dominoes”; therefore it is the answer for this problem too. Some children remembered that such problems are called isomorphic (page 29). ■

Problem 3.3. Parking Cars Only. A parking lot has 10 narrow parking spaces in a row. Each car takes two spaces. How many ways are there to park cars in this parking lot if some parking spaces might remain empty? What if there are 12 parking spaces?

Unexpectedly, a few kids could not visualize parking arrangements with empty spaces. To help them we had to discuss a few examples, drawing them on the board.

As with the previous problem we suggested the children investigate shorter parking lots: “What if there are only two or three parking spots? Can you draw the possible parking arrangements?” Using this hint most

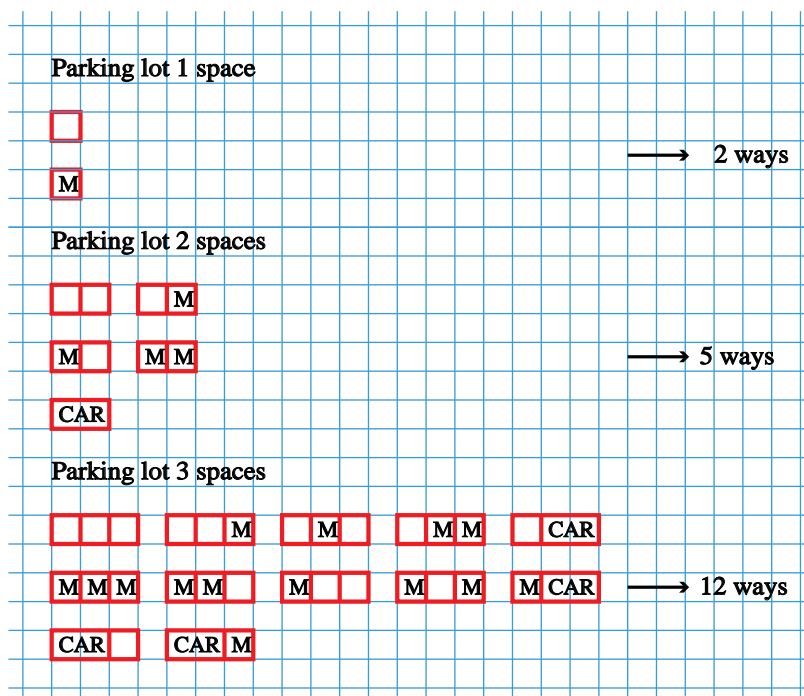
of the children easily drew the pictures for lengths 2, 3, and 4, however some students omitted the case of a completely empty parking lot. When this mistake was noticed most of the class was able to explain how to turn this problem into the previous one: park a motorcycle in each empty space. Several students also suggested to translate the word “motorcycle” to “free space”.

Thus, the answers to the problem are 89 and 233. ■

Problem 3.4. Parking Cars and Motorcycles With Empty Spaces.

A parking lot has seven narrow parking spaces in a row. A motorcycle takes one space, and a car takes two spaces. How many ways to park cars and motorcycles are there if some parking spaces might remain empty?

This problem is harder than the ones discussed before. Once again we advised our students to start with the simpler examples: parking lots with 1, 2, or 3 parking spaces. The results are shown below:



Many children remembered a similar system of listing strips in Problem 3.1, “Squares and Dominoes”, and gave the following recipe:

- List all parking lots with the first space empty.
- Then list all the parking lots with a motorcycle in the first space.
- Then list all the parking lots where the car is parked in the first two spaces.

Several kids could proceed on their own, the rest needed our hint: “Why does the first row have five pictures, while the last row has two?” The hint

helped and many children were able to explain what would happen with four parking spaces:

- (1) If the first space is empty, what remains has three spaces. We already know that there are 12 ways to park in the parking lot with three spaces.
- (2) If the first space is taken by a motorcycle, what remains has three spaces. Again, there are 12 ways to park in this case.
- (3) Finally, if there is a car on the left, then what remains has two spots. As we saw there are five ways to park in these spots.

There are no other possibilities of what can be parked in the first spot. The total number of parking arrangements for the parking lot of size four is:

$$12 + 12 + 5 = 2 \times 12 + 5 = 29.$$

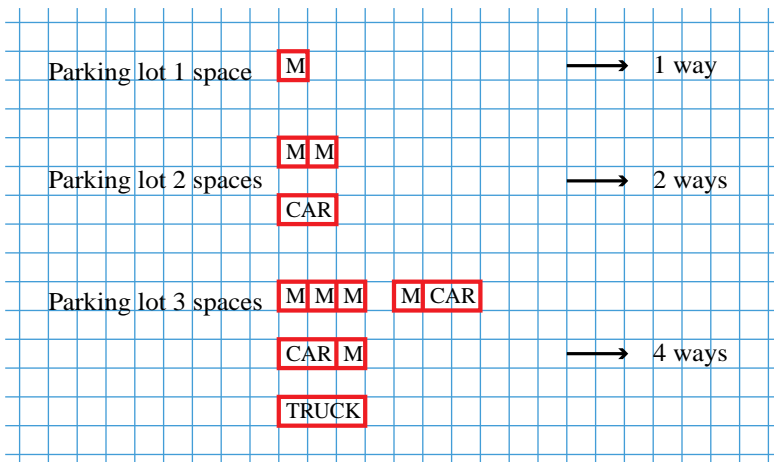
The same reasoning can be used for larger parking lots. Going from the parking lot with four spaces to a parking lot with five spaces, from the parking lot with five spaces to a parking lot with six spaces, and so on, the kids came up with the table:

Number of parking spots	1	2	3	4	5	6	7
Number of ways to park	2	5	12	29	70	169	408



Problem 3.5. Parking Cars, Motorcycles, and Trucks. A parking lot has 10 narrow parking spaces in a row. A motorcycle takes one space, a car takes two spaces, and a truck takes three spaces. How many ways to park cars, motorcycles, and trucks are there so that all the spaces are taken?

The class began the problem by looking at shorter parking lots. The children easily drew all possible parking arrangements for the parking lots of length 1, 2, and 3:



Then the students quickly came up with the solution for a parking lot with four spaces:

- (1) If there is a motorcycle in the first spot, what remains has three spaces. As we saw there are four ways to park in these three spaces.
- (2) If there is a car on the left, what remains has two spaces. We already know that there are two ways to park in these two spaces.
- (3) If there is a truck on the left, what remains has one space. We already know that there is only one way to park in one space.

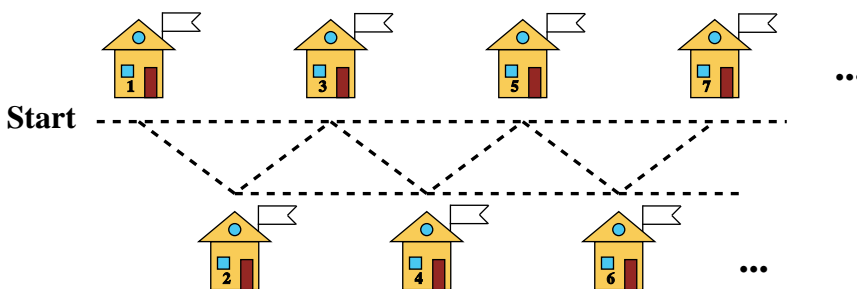
Since we listed all the possibilities for the type of vehicle at the left end, the total number of parking arrangements is $1 + 2 + 4 = 7$. Continuing in the same manner for longer parking lots, the kids filled the following table:

Number of parking spots	1	2	3	4	5	6	7	8	9	10
Number of ways to park	1	2	4	7	13	24	44	81	149	274

Counting Routes

This is yet another problem where the answer is a Fibonacci sequence.

Problem 3.6. Postman Rob. In a village, postman Rob always delivers mail going left-to-right horizontally or diagonally:



Depending on who gets letters today, he may skip some houses. In how many ways can Rob reach houses #1, #2, #3, #5, and #10?

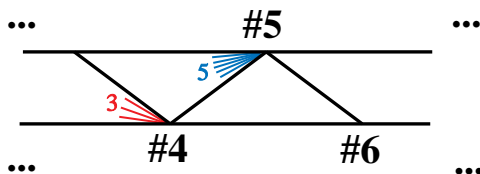
To help the kids to organize their answers we recommended recording the number of ways to the house on its banner. Once a few kids who missed the diagonal way to house #3 through house #2 were corrected, everybody had the answers 1, 1, 2, 3, and 5 on the banners of houses #1, #2, #3, #4, and #5. At this point the class recognized the Fibonacci sequence, but very few kids were able to explain why the pattern will continue.

It took a few more minutes and tracing all the paths to house #6 for the majority of the class to discover that Rob can get to house #6 only from house #4 or from house #5. While many children could immediately replace various paths to house #4 (or #5) by their count, the rest continued to think

of them as a collection of pictures. The latter group would not recognize the importance of the discovery above even if they made it. The following hints helped most of the class to finish the problem:

“What could be the last house Rob passed on his way to house #6?”

“Observe the picture below, where the blue and the red lines denote the ends of the various paths to houses #4 and #5. Draw colored dashed lines to make it clear that the number of paths reaching house #6 is $3 + 5$.”



Some children drew the picture on the left, others insisted that the picture on the right with multiple dashed lines is more clear:



Finally, everybody was able to see that the same rule, “the next is the sum of two previous”, holds true for all houses. We got the Fibonacci sequence again! There are 55 ways leading to house #10. ■

Fibonacci Sequence in Nature

The answers to many of the preceding problems form one of the most well-known number sequences, the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, \dots$$

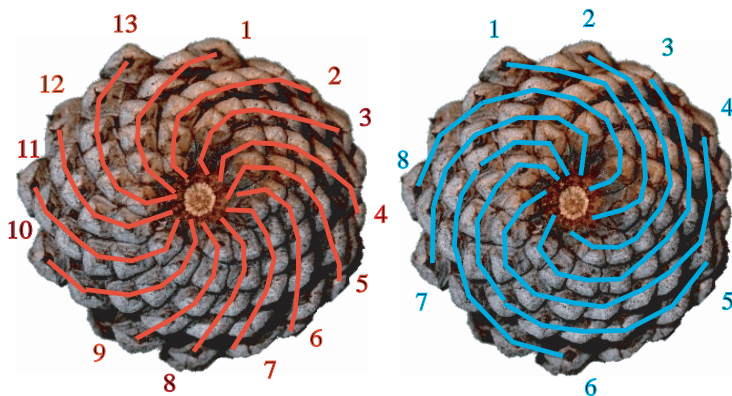
It was first described over two thousand years ago; the earliest records of this sequence go back to the 2nd century BCE in India.

The sequence is called Fibonacci in honor of Leonardo Fibonacci, an Italian mathematician who introduced it to the Western World in 1202 in his book *Liber Abaci*, which means “Book of Calculation”.¹

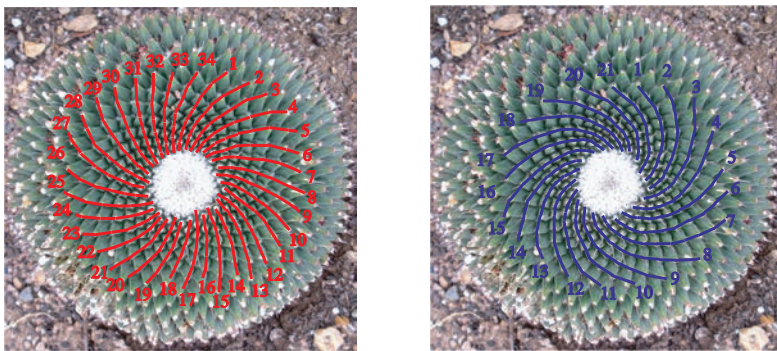
Math Context. The typical notation for Fibonacci numbers is F_n . The defining recurrence relation is $F_n = F_{n-2} + F_{n-1}$, with starting values $F_1 = 1, F_2 = 1$.

¹Handout about Fibonacci’s life is on page 167.

Fibonacci numbers appear not only in mathematical contexts — they are often discovered in nature. For instance, the numbers of spirals in many naturally occurring spiral formations turn out to be Fibonacci numbers. They appear in pine cones, like the one shown below with 13 spirals going clockwise and eight spirals going counterclockwise. Observe that 8 and 13 are two consecutive Fibonacci numbers.



Cacti provide another example. Examining the spirals of the cactus below, one can observe 34 spirals going clockwise and 21 spirals going counterclockwise. Again, 21 and 34 are two consecutive Fibonacci numbers.



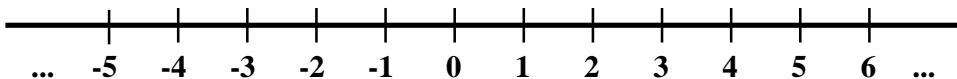
In reality when one picks up a random pine cone or a cactus, the spirals may not be as perfect as in the examples above. This happens when the plants get sick or their growth is constrained. It is cheating to count spirals when some are incomplete or to draw fictitious curves. In class we helped the children to confirm that all the red spirals on the left picture are present as whitish curves on the right picture for both cones and cacti. So the kids could see that we had no need to cheat.

Even larger Fibonacci numbers are sometimes found in nature. In sunflowers the number of spirals can be as high as 89 in one direction and 144 in the other.

Although Fibonacci numbers in nature were observed in many ancient cultures long before Fibonacci's time, the general cause of this phenomenon is still debated by modern scientists.

Extension to the Left

Is it possible to extend the Fibonacci sequence to the left? Before we can answer this question one needs to know how to add negative numbers. Many students have not studied negative numbers yet, although most have heard about them. So, we drew the number line:



In our experience kids have absolutely no issue with negative numbers appearing on the number line to the left of 0. To expose the kids to the arithmetic with negative numbers we introduced a grasshopper which jumps between numbers in steps of 1. First we placed the grasshopper at 2 on the number line. What should the grasshopper do to model adding 3, 2, 1, or 0 to 2? After getting the answers, we asked the same question about adding -1 , or -2 , or -3 . The class followed the numeric (or pictorial) pattern and concluded: adding a positive number is “moving right”, adding 0 is “staying at the same place”, and adding a negative number is “moving left”. Surprisingly, the children had no problem giving instructions to the grasshopper when it starts at -2 .

Problem 3.7. Fibonacci Numbers to the Left of 1, 1. Extend the Fibonacci sequence to the left, so that the numbers still follow the Fibonacci rule: every number is the sum of the two preceding numbers.

A few students fluent with subtraction of negative numbers were able to quickly finish the problem. We helped the rest with the leading questions, starting with, “What should we put instead of ‘?’ in the sequence ?, 1, 1, 2, 3, 5 and why?” Students immediately answered 0. Out of various explanations we selected the one that involved solving $? + 1 = 1$.

“What is ‘?’ in ?, 0, 1, 1, 2, 3, 5 and why?” No one had any doubts that it is 1, and the majority explained it using $? + 0 = 1$.

At this moment many students could proceed by themselves, the rest needed more leading questions.

Many children found “?” in ?, 1, 0, 1, 1, 2, 3, 5 with no difficulty by solving $? + 1 = 0$. Only a couple of students needed a hint, “Where should we place the grasshopper to find ‘?’ in $? + 1 = 0$?”

Finding “?” in the sequence ?, -1 , 1, 0, 1, 1, 2, 3, 5 led to a heated discussion. The kids with the correct answer 2 managed to convince the rest of the class using the grasshopper in their arguments.

Soon everybody arrived at the extension $\dots, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, \dots$.

“Can you see a pattern?” — “Yes!” The numbers on the left of zero are the same as on the right of zero except for their signs. One can imagine that an enchanted mirror is placed at zero. It preserves the magnitude of the numbers but makes their signs alternate. ■

At this time we explained to the class that finding “?” on each step of the above problem can be done via subtraction. For example, finding “?” in $? + 1 = 1$ is the same as calculating $1 - 1$. Almost everybody knew it; the kids explained, “Subtraction inverses addition.” To test them we asked, “What should the grasshopper do to describe this subtraction?” Again, nobody had doubts that it should start at 1 and jump one step to the left. Several kids pointed out that in addition and subtraction the grasshopper moves in opposite directions.

Now, the class was ready to consider a much more confusing case: subtracting a negative number. When solving $? + (-1) = 1$, we found that $? = 2$. With subtraction, we write it down as $1 - (-1) = 2$. “What should the grasshopper do to model this subtraction?” This led to a short discussion and after a few more examples the class concluded: on the number line, subtracting a positive number “is moving left”, subtracting 0 “is staying at the same place”, and subtracting a negative number “is moving right”. These directions are opposite to those in addition, which makes sense since one can undo addition using subtraction.

Math Context. The Fibonacci sequence may be extended to negative numbers using $F_{n-2} = F_n - F_{n-1}$. So we get:

$$\dots, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, \dots;$$

$$\text{moreover } F_{-n} = (-1)^{n+1}F_n.$$

Even/Odd Pattern

Analyzing the even/odd pattern of the Fibonacci numbers gave us an opportunity to discuss which numbers are *odd* and which are *even*.

In our classes we observed that the children were taught three different definitions of even numbers:

- (1) A number is even if it can be divided into pairs (or can be obtained via “counting by two”).
- (2) A number is even if it can be split into two equal parts.
- (3) A number is even if its last digit is 0, 2, 4, 6, or 8.

In our math circles we work only with the first two definitions. Their equivalence can be easily explained using socks. The number of socks is even if they can be counted by two or paired up. If we now break all pairs putting

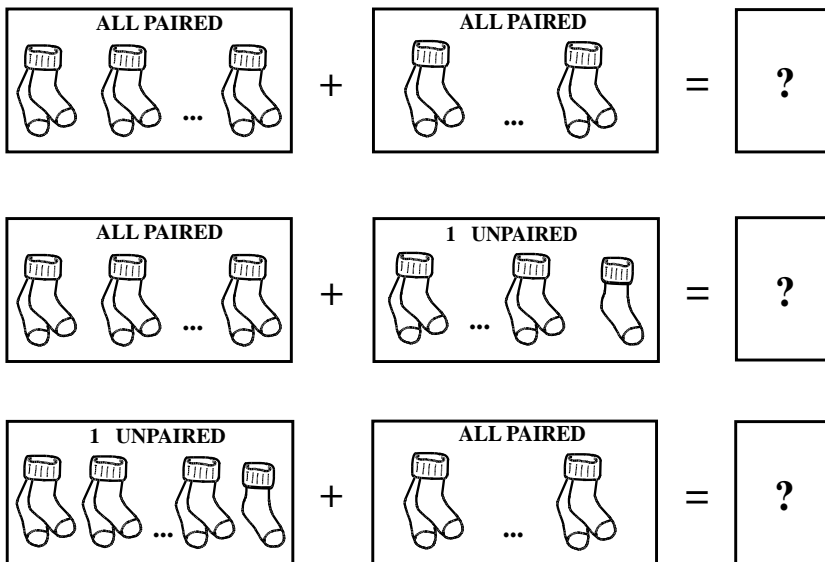
one sock into the first pile and the other sock into the second pile, we end up splitting the number of socks into two equal parts. This process can be reversed: socks divided into two equal piles can be paired up by taking one sock from the first pile and another sock from the second pile.

For us the third approach is not a definition, but a shortcut to checking whether a number is even. Why does it work? Only a few children could explain without our prompts that any number can be broken into “tens and ones”. For example, 435 is 43 tens and 5 ones. Tens can always be broken into two equal parts since $10 = 5 + 5$, so only the last digit matters.

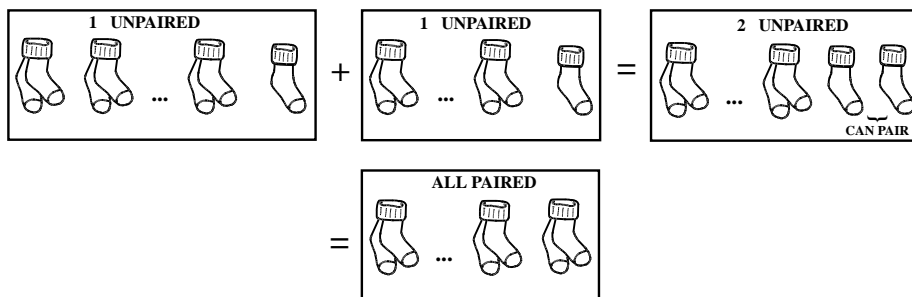
What numbers are odd? Some students tried to formulate a negation of the sentence that defines “even”: “Odd numbers cannot be split into two equal parts.” Instead, we challenged the class, “Now, when you know what is ‘even’, you can define ‘odd’ using fewer words.” After a couple of attempts the children arrived at, “Odd means not even.”

At this point we needed to establish the “even + even”, “even + odd”, “odd + even”, and “odd + odd” rules. Many students knew these rules but could not explain them.

We like to verify these rules using numbers represented by boxes with socks. We call it the “socks in a box” approach. If the number is even, the socks in the box are paired. If the number is odd, there is one unpaired sock left in the box (this happens if the sock missing from the pair was eaten by a “sock monster”). Combining different boxes gives us the desired rules:



Nobody in our class had any doubt what the results would be in the cases above. Most could also figure out what happens in the last “odd + odd” case. Some students, however, needed to draw an additional picture with two unpaired socks:



To summarize:

$$\text{EVEN} + \text{EVEN} = \text{EVEN},$$

$$\text{EVEN} + \text{ODD} = \text{ODD},$$

$$\text{ODD} + \text{EVEN} = \text{ODD},$$

$$\text{ODD} + \text{ODD} = \text{EVEN}.$$

Now we were ready to investigate *parity* (property of numbers to be even or odd) in the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$$

The students noticed that numbers in the sequence go in the following order: odd, odd, even, odd, odd, even, . . .

Problem 3.8. Even/Odd Pattern. Why does the parity of the Fibonacci numbers repeat in the pattern “odd, odd, even” again and again?

Teacher → This problem is more difficult than typical problems in this book and we guide the children by posing leading questions.

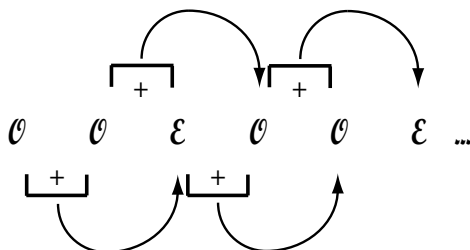
Let’s look at the parity of the first few Fibonacci numbers. We use letters “ \mathcal{O} ” and “ \mathcal{E} ” for even and odd to avoid confusion of O and 0 :

1	1	2	3	5	8	13
↓	↓	↓	↓	↓	↓	↓
\mathcal{O}	\mathcal{O}	\mathcal{E}	\mathcal{O}	\mathcal{O}	?	

Number 8 is even and so the “?” is \mathcal{E} . Let us cover the framed numbers, 8 and 13, by a piece of paper. Can we still find out whether “?” is even or odd without calculating the sum $3 + 5$? The children immediately shouted “Yes” and provided an explanation: to the left of the “?” we can see $\mathcal{O}\mathcal{O}$ (3 and 5 are both odd). Any $\mathcal{O}\mathcal{O}$ is followed by \mathcal{E} , since $\text{ODD} + \text{ODD} = \text{EVEN}$. Likewise, combinations of \mathcal{O} and \mathcal{E} , either $\mathcal{O}\mathcal{E}$ or $\mathcal{E}\mathcal{O}$, will be followed by \mathcal{O} .

Now, instead of covering 8 and 13, let us cover all numbers after 1, 1. Then all we know about the second row is that it starts with $\mathcal{O}\mathcal{O}$.

Can the second row be continued as was done before? The children came up with the following picture:



How can we be sure that this pattern continues forever? Many children needed another hint: “When we see OOE somewhere, what will follow?” Some students found the answer right away; others had to continue the pattern for a little longer. The class concluded: we do not care whether OO starts the Fibonacci sequence or if OO appears anywhere in the sequence, it will always turn into $OOEOOE$. So, when we see OOE it will be followed by OOE again and again and again! The pattern will repeat forever if it appears once, and we know that it appears at the start. ■

Divisibility by 3

Teacher → This section is intended for elementary school students who have already studied division.

Let’s look at the Fibonacci sequence again, coloring numbers divisible by 3 in red, and numbers not divisible by 3 in blue:

$$1 \ 1 \ 2 \ 3 \ 5 \ 8 \ 13 \ 21 \ 34 \ 55 \ 89 \ 144 \ \dots$$

As one can see, every fourth number is divisible by 3 (colored in red).

Problem 3.9. Divisibility by 3. Explain why every fourth number in the Fibonacci sequence is divisible by 3.

Several children suggested using a strategy for RED/BLUE numbers similar to the one used for ODD/EVEN numbers in the previous problem.

We decided to check if such a strategy works for the first few Fibonacci numbers:

$1 + 1 = 2$	$BLUE + BLUE = BLUE$
$1 + 2 = 3$	$BLUE + BLUE = RED$
$2 + 3 = 5$	$BLUE + RED = BLUE$
$3 + 5 = 8$	$RED + BLUE = BLUE$

“Can we say something definite about the color of BLUE+BLUE?” — “No!” Looking at the top two rows we see that the result can be different. Apparently, with the divisibility by 2 (EVEN/ODD) we were lucky that “ODD + ODD” is always “EVEN”, so we could write the rule for addition:

“ODD + ODD = EVEN”. The same approach does not work for divisibility by 3:

NON-DIVISIBLE-BY-3 + NON-DIVISIBLE-BY-3

may be of both types, divisible by 3 or not.

That means we have to do more work before we can proceed with the problem. We come back to this problem on page 58 towards the end of this section.

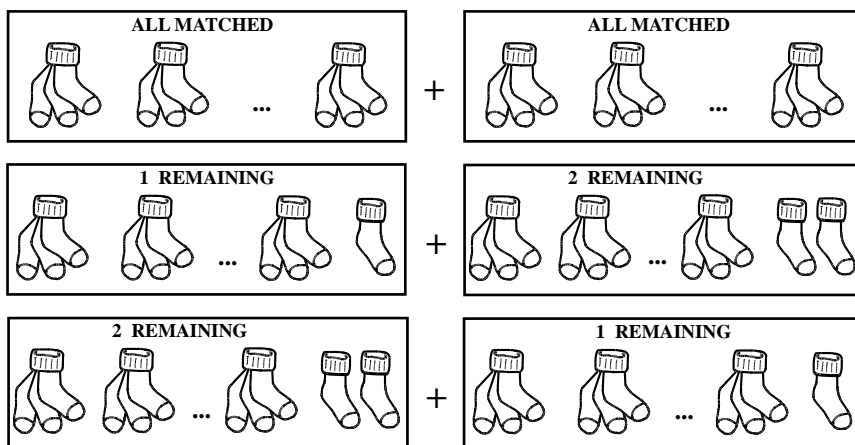
Let us use the “socks in a box” approach again. Consider a family of aliens from Mars who join their socks by threes. “Why by threes?” The children immediately replied, “Obviously because aliens from Mars have three legs!”

Martians keep their socks in boxes and join as many socks in every box as they can into triples. Nevertheless, some socks may remain unjoined — sock monsters eat socks even on Mars! What happens if one combines two such boxes?

Problem 3.10. Triplets of Socks. In a Martian family everybody joins as many socks as possible into triples. Grandpa has a box of socks. Baby has a box of socks too. After combining their socks into one box, all socks may be joined by 3. How could the socks in Grandpa’s and Baby’s boxes look before they were combined:

- If all Grandpa’s socks were joined, what about Baby’s socks?
- If Grandpa’s box had unjoined socks, how could Baby’s box look?

The students realized that there are two separate “Grandpa has unjoined socks” cases: Grandpa has one single sock, or he has two single socks. After that the children quickly came up with the list of three possible situations:



We recorded these results as:

$$0_R + 0_R = 0_R,$$

$$1_R + 2_R = 0_R,$$

$$2_R + 1_R = 0_R.$$

In these equalities 0_R represents a box with zero unjoined socks, 1_R a box with one unjoined sock, and 2_R a box with two unjoined socks. ■

Problem 3.11. Addition of the Remainders. We already have some entries in the following addition table:

+	0_R	1_R	2_R
0_R	0_R		
1_R			0_R
2_R		0_R	

Fill in the empty cells.

Some of our students filled the empty cells in the table very quickly, while others didn't know how to begin. We suggested imagining or drawing pictures with socks for each cell. A couple of students still struggled, and we helped them to draw a picture for the cell $2_R + 2_R$. In a few minutes everyone filled in the addition table modulo 3:

+	0_R	1_R	2_R
0_R	0_R	1_R	2_R
1_R	1_R	2_R	0_R
2_R	2_R	0_R	1_R

This is an appropriate time to tell the students that the symbols 0_R , 1_R , and 2_R are called *remainders* in math. For example, 11 socks in a box will be joined as $3 + 3 + 3 + 1 + 1$. So, the two unjoined socks “remain” and we say that the remainder is 2 when one divides 11 into groups of 3. ■

Problem 3.12. Remainders Modulo 3 for Fibonacci Numbers. Replace every Fibonacci number by a box with that many Martian socks. In every box as many socks as possible are joined into triplets. Label every box as 0_R , 1_R , 2_R depending on the number of unjoined socks. Find the pattern in the obtained labels.

Hint: “When labeling a box, do not divide the number of socks by 3; instead, combine the two previous boxes, using the rule for Fibonacci numbers.”

Some children still started dividing the Fibonacci numbers by 3. Unfortunately this approach is error-prone and cannot be easily extended to the larger numbers. So we asked the students to switch to the approach suggested by the hint.

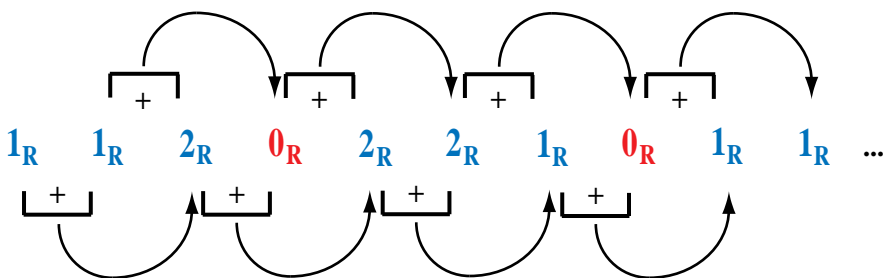
With the help of the class we wrote the first few remainders on the board:

- The first two numbers are 1, 1, so we wrote $1_R, 1_R$.
- The next number 2 is $1 + 1$; combining two boxes of type 1_R results in a box of type 2_R . Instead of directly finding the box type for number 2, one may just look at the types of two boxes to the left of it!
- The next box type is obtained by combining a box of type 1_R with a box of type 2_R . It is $1_R + 2_R = 0_R$, as shown in the table created in the previous problem.

1	1	2	3	...
1_R	1_R	2_R	0_R	...

A couple of students thought that the sequence would repeat itself in blocks $1_R, 1_R, 2_R, 0_R$. We asked these children to write a few more terms of the sequence to confirm their hypothesis.

Pretty soon the group obtained the following:



Some children stopped writing the sequence after $1_R, 1_R, 2_R, 0_R, 2_R, 2_R, 1_R, 0_R, 1_R, 1_R$ saying that $1_R, 1_R$ at the end means that the sequence is going to repeat itself. Others wrote the sequence further, and after six more additions discovered the repeating pattern. ■

Now, let us go back and finish Problem 3.9.

The concluding argument is practically the same as in the even/odd problem, but, as usual, for many children it was hard to articulate why the pattern observed in the previous problem will continue forever. Whenever we see the combination $1_R, 1_R$ (whether at the start of the Fibonacci sequence or anywhere else) it will be followed by $2_R, 0_R, 2_R, 2_R, 1_R, 0_R$. Then it will be followed by $1_R, 1_R$ again. It means that whenever we see $1_R, 1_R, 2_R, 0_R, 2_R, 2_R, 1_R, 0_R$ (as a part of the sequence), it is going to be followed by the same part again! Since this part of the sequence appears at the beginning, it will repeat forever.

Inspecting the above sequence, we see that every 4th entry is 0_R , which means that every 4th Fibonacci number is divisible by 3.

This finishes discussion of Problem 3.9. ■

Several children asked whether we would see a repeating pattern if we color the Fibonacci numbers divisible by a number different from 2 or 3. We replied, “Yes, but it is a hard problem.”

Math Context. The more general statement concerning divisibility of Fibonacci numbers is that if $k|n$, then $F_k|F_n$. Moreover, for every number m , some of F_k are going to be divisible by m . Taking the smallest such k , one may conclude that every k th number is divisible by m .

Sum of the First n Consecutive Fibonacci Numbers

For the sequence of Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, \dots , consider the “*running sums*” 1, $1 + 1$, $1 + 1 + 2$, $1 + 1 + 2 + 3$, $1 + 1 + 2 + 3 + 5$, $1 + 1 + 2 + 3 + 5 + 8$, \dots .

Problem 3.13. Running Sums of the Fibonacci Numbers. Calculate the first few “running sums”. Can you find the pattern in the answers? Can you show that the pattern will continue forever?

We began by reminding our students that real mathematicians often make discoveries by first observing a certain pattern, and then finding an explanation for it. This problem asks us to do the same.

We suggested the children use a table for writing the running sums. They quickly filled the two left columns shown below. It took them a couple more minutes to notice that the running sums may be written as a Fibonacci number minus 1 (recorded in the right column):

Running Sum of Fibonacci numbers	Total	Fibonacci number -1
1	1	2 $- 1$
$1 + 1$	2	3 $- 1$
$1 + 1 + 2$	4	5 $- 1$
$1 + 1 + 2 + 3$	7	8 $- 1$
$1 + 1 + 2 + 3 + 5$	12	13 $- 1$
$1 + 1 + 2 + 3 + 5 + 8$	20	21 $- 1$

What is the explanation? Will the pattern continue?

Hint: “Instead of subtracting 1 in the right column, let us add 1 in the left column.” Now the “Total” column of the updated table contains Fibonacci numbers. We need to show that if we continue the table, the numbers in the right column will always be Fibonacci numbers.

1 +	Sum of Fibonacci numbers	Total
1 +	1	2
1 +	1 + 1	3
1 +	1 + 1 + 2	5
1 +	1 + 1 + 2 + 3	8
1 +	1 + 1 + 2 + 3 + 5	13
1 +	1 + 1 + 2 + 3 + 5 + 8	21

Hint: “Look at the entries in the left columns of the table above. How do they change from row to row?” The kids quickly noticed that they differ only by the last term. Let us modify the table once more. In the left column we put parentheses around the terms that repeat the previous row (in green below). Now all the children realized that the sums in parentheses are already computed; they are “Totals” in the previous row (shown in red). This observation is recorded in the column “Or”.

1 +	Sum of Fibonacci numbers	Or	Total
1 +	1		2
(1 + 1)	+ 1	2 + 1	3
(1 + 1 + 1)	+ 2	3 + 2	5
(1 + 1 + 1 + 2)	+ 3	5 + 3	8
(1 + 1 + 1 + 2 + 3)	+ 5	8 + 5	13
(1 + 1 + 1 + 2 + 3 + 5)	+ 8	13 + 8	21

Thus the green numbers in the middle column (column “Or”) are the sums of the numbers in parentheses on the left. At the same time they are copies of the red numbers above on the right (column “Total”). Suppose we already know that a certain red number (for example **13**) is a Fibonacci number. Then the green number in the next row is also a Fibonacci number, and so the sum in the middle column ($13 + 8$) is a sum of two consecutive Fibonacci numbers. Since the sum of two consecutive Fibonacci numbers is the next Fibonacci number, the next red number (**21**) must also be a Fibonacci number. This pattern will continue forever! ■

Teacher → This argument lacks an explanation of why the numbers added in the column “Or” are consecutive Fibonacci numbers. Children just stated this fact, but did not notice that they need to prove it. We decided not to focus their attention on this gap.

Math Context. $\sum_{i=0}^n F_i = F_{n+2} - 1$. This fact is easily proved by induction.

Fibonacci Rectangles and Fibonacci Spiral

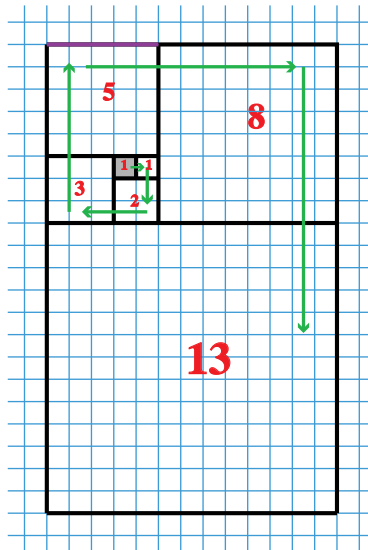
Problem 3.14. A Mathematical Fairy Tale.

- In the middle of a beautiful forest an Ant builds a house. The house has one square room (the small grey square in the picture below).
- A Ladybug loves the place and wants to live there too; so she extends the house by attaching another square room on the right (the small square to the right of the first one).
- A Snail comes, loves the place, and attaches another square room for herself. Since the Snail is larger, she attaches the room on the bottom.
- This process is repeated by the Mouse, who attaches a square room on the left.
- The house extension is continued by a Rabbit, a Fox, a Lion, and so on.

Extensions are attached in the clockwise direction shown by the arrows: first \rightarrow ; then \downarrow ; then \leftarrow ; and so on.

(a) Show that if we continue adding larger and larger rooms for the creatures, the sides of the rooms will always be Fibonacci numbers.

(b) Show that after each addition the sides of the house will also form a Fibonacci sequence.



Before starting this problem we asked the children a couple of leading questions.

“What is the shape of the house after each addition?” It’s a rectangle. The first house, the Ant’s, is a square but squares are also rectangles.

At this moment we discussed what a rectangle is and listed its properties on the board. One of these properties, the equality of opposite sides, is essential for the solution.

“Can you draw the next room, which belongs to a Hippo?” Almost everyone in the class drew the next square with side 21 without any problem. All the students were sure that the sides of the rectangles form the Fibonacci sequence, but most could not explain why.

Let us look at the side of the largest square room in the picture above, 13. The top side is the sum of the sides of the three smaller squares: $3 + 2 + 8$. However, Fibonacci numbers are formed by a different rule: they are the sum of two (not three) previous numbers! To follow the Fibonacci sequence the sides of the squares should be equal to the sum of the sides of the two previous squares, $13 = 5 + 8$. The 8×8 square is next to the 13×13 square, but the 5×5 square is not, and so several children needed a hint: “On the picture mark the part which illustrates the addition of $5 + 8$.” After marking this on the top side of the picture, almost everybody noticed that the segments of length 3, 2, and 8, and the segments 5 and 8 form two opposite sides of the 8×13 rectangle (the house after the Fox’s addition). Hence they have the same length!

On every step of the house expansion the longer side of the rectangle is the sum of the sides of the two largest rooms. This means that every new square has a side equal to the sum of the sides of the two previous squares. Hence, the sides of the squares form a Fibonacci sequence.

It wasn’t very hard for the students to notice that the smaller side of the rectangle is equal to the side of the last added square, and the larger side to the side of the next square-to-add. Since the sides of the squares form the Fibonacci sequence, the sides of the rectangles also form a Fibonacci sequence. ■

These rectangles (the houses) are called *Fibonacci rectangles*.

Problem 3.15. Running Sums of Squares of Fibonacci Numbers.

Observe:

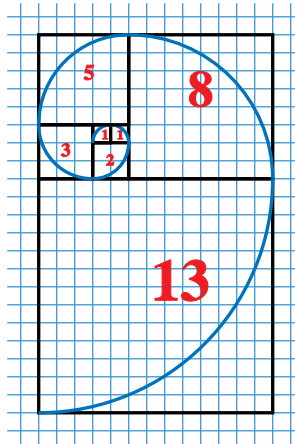
$$\begin{aligned} 1^2 + 1^2 &= 1 \times 2, \\ 1^2 + 1^2 + 2^2 &= 2 \times 3, \\ 1^2 + 1^2 + 2^2 + 3^2 &= 3 \times 5, \\ 1^2 + 1^2 + 2^2 + 3^2 + 5^2 &= 5 \times 8. \end{aligned}$$

Will the pattern continue? Why?

Hint: “Remember the Fibonacci rectangles.”

The majority of the class did not compute the sum on the left or the product on the right. Instead they identified the summands on the left and the factors on the right with the house from the Mathematical Fairy Tail. At each step of the house extension the area of the whole house is the sum of the areas of the Fibonacci squares (individual rooms). This verifies the equality, and the pattern will continue forever! ■

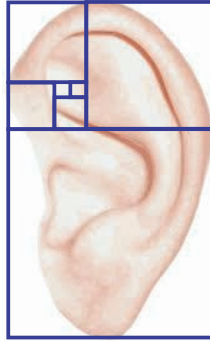
We finished the discussion of the Fibonacci rectangles with a couple of interesting curves. If we draw a quarter of a circle in each square, we get a spiral-like blue curve:



If we had a magnifying glass we would see the “ugly” junctions at the ends of the quarter-circles. There is a way to change the quarter-circles into a better-matching curve called the *Fibonacci log-spiral*. This log-spiral looks “smoother” but with the help of a magnifying glass one could see tiny discrepancies: it does not go exactly through the corners of the squares. The smaller the square, the larger the discrepancy. Still, it is really amazing how good the match is for the larger squares.

Teacher → The log-spiral is much “smoother”, but nevertheless does not go exactly through the corners of the squares. Our children are too young to understand what a Fibonacci logarithmic spiral is. The Fibonacci sequence matches a geometric progression with ratio φ well, but it doesn’t match it exactly.

The log-spiral is often seen in nature; for example, snail shells often fit the Fibonacci log-spiral. However, a lot of such coincidences are just wishful thinking. For example, some people claim that the curves of a human ear match the Fibonacci log-spiral.



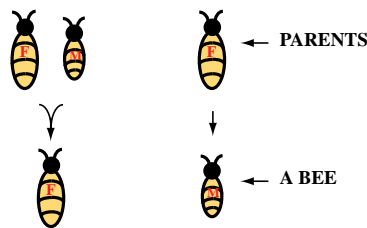
Can you find where the flaws are in this match? The children greatly enjoyed finding them, although some needed our help: “Look at the corners of the squares. On which curve of the ear are they positioned?” The corners of the squares lay on different spirals.

Honeybees’ Ancestral Tree

Leonardo Fibonacci obtained his sequence modeling the procreation of rabbits. The following problem is about honeybees’ ancestry tree. It is also biological, but more realistic and much easier to formulate.

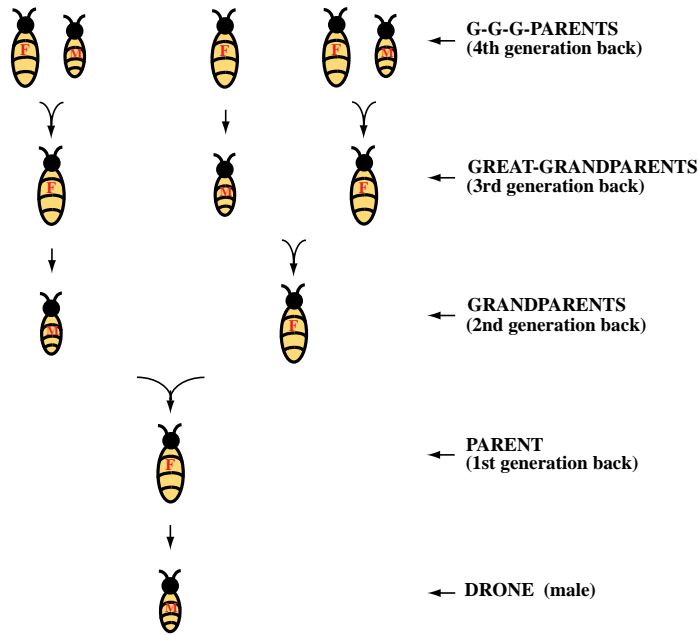
Ancestry trees, unlike family trees, show only direct ancestors: parents, grandparents, great-grandparents, and so on, but they do not show siblings. In an ancestry tree the parents can be drawn above or below their child. We will draw them above.

Female honeybees have two parents, a male and a female, whereas male honeybees (drones) have just one parent, a female:



Problem 3.16. Small Ancestry Tree. Draw an ancestry tree of a drone. Go back to the 4th generation.

The children promptly produced the following picture:



Problem 3.17. Ancestors of Honeybees. How many ancestors does a drone have in generations 5, 6, and 10?

Teacher → This problem is complicated, so we discuss it at the end of the chapter, guiding students with a lot of hints.

First, the class counted bees in each generation in the above picture and was amazed to discover a familiar sequence: 1, 1, 2, 3, 5. The students understood that they need to explain that the next number in the sequence is the sum of the previous two, but had no clue why. Hint: “Count males and females in each generation separately and record the results in a table.” The results are:

Generation (back)	0	1	2	3	4	5	6	7
Number of male ancestors	1	0	1	1	2			
Number of female ancestors	0	1	1	2	3			
Total number of ancestors	1	1	2	3	5			

“Can you draw a bigger ancestry tree and fill in the remaining columns of the table?” The students like drawing the big ancestry trees and had no problem with getting the answers, shown in the next table. They observed that the number of male ancestors, female ancestors, and the total number of ancestors form Fibonacci sequences. However, the sequence for the male ancestors begins with 1, 0, 1; for the female ancestors it begins with 0, 1, 1; and for the total amount of ancestors it begins with 1, 1, 2. So, there are shifts in the Fibonacci sequence in the table.

Generation (back)	0	1	2	3	4	5	6	7
Male	1	0	1	1	2	3	5	8
Female	0	1	1	2	3	5	8	13
Total	1	1	2	3	5	8	13	21

Why do all the rows in the table above form Fibonacci sequences?

Hint: “Look at the repeating numbers in the diagonals of the table and try to explain the equality. For example, why is the number of female bees in the 5th generation back equal to the total of bees in the 4th generation back?” Some of the students came up with the explanations similar to: “It’s an ancestry tree, so each female has a child, and the number of children is displayed on the left in the ‘Total’ row.”

Now a couple of students anticipated our next question and voiced it on their own, “Why is the number of males in the 6th generation back equal to the number of females in the 5th generation back?” While this question was harder, the children were in the right mindset, “Every male in the ancestry tree has one daughter, and the daughters are displayed on the left.” There is one exception to this claim, the first drone child, but we did not mention it in class.

Now we have an explanation of the diagonal repetitions in the table. Is it possible to use them to find “?” in the table below?

Male	1	0	1	1	...				
Female	0	1	1	2	...				
Total	1	1	2	3	...	34	55	?	

The students knew that it should be “89” but it was just a guess, based on continuing the Fibonacci sequence. We required an explanation and gave a hint: “Think diagonally!” The children filled the table, using the already explained diagonal patterns:

Male	1	0	1	1	...			34	55
Female	0	1	1	2	...		34	55	
Total	1	1	2	3	...	34	55	?	

Look at the column with red numbers 34, 55, and “?”:

- The red 34 in the top row is the count of males; in the bottom row black 34 stands for the total number of bees 2 generations prior to “?”.

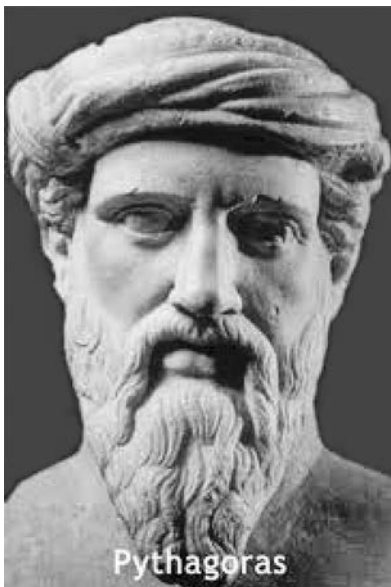
- The red 55 in the second row is the count of females; in the bottom row black 55 stands for the total number of bees 1 generation prior to “?”.

The total number of bees in a certain generation is the sum of females and males in that generation. So, finding the total we sum up numbers circled in red; but instead we can sum up numbers circled in green:

		34
		55
34	55	?

Therefore, we end up with the Fibonacci sequence, where any number starting with the third is the sum of two previous numbers. Hence, the drone has eight ancestors in the 5th generation back, 13 ancestors in the 6th generation back, and 89 in the 10th generation back. ■

Handouts



Pythagoras (570 BCE – 495 BCE)

Pythagoras is often referred to as the first pure mathematician. The exact dates of his birth and death are not known. Various records show that he was born on the island of Samos, Greece, in approximately 569 BCE, and died sometime between 500 BCE and 475 BCE in Metapontum, Italy.

Pythagoras was well educated, he played the lyre throughout his lifetime, knew poetry, and recited Homer. He was interested in mathematics, philosophy, astronomy, and music.

Pythagoras settled in Crotona, a Greek colony in southern Italy, and founded a philosophical and religious school where many of his followers lived and worked. The Pythagoreans' beliefs were based on the power of numbers, honesty, living a simple, unselfish life, and trying to show kindness to people and animals. The followers of Pythagoras were known as *mathematikoi*.



Carl Friedrich Gauss (1777 – 1855)

Carl Friedrich Gauss is sometimes called the “Prince of Mathematics” or the “greatest mathematician since Ancient Greece.”

Carl Gauss, a true child prodigy, was born into a poor, uneducated family in Brunswick, Germany, on April 30, 1777. Gauss amazed his parents by learning to add numbers and make calculations before he was able to talk. When he was just three years old, he, supposedly, was already correcting his father’s account books. The Duke of Brunswick heard of Gauss’s outstanding abilities and gave the 15-year old boy a stipend to study at the university. There, he began making new mathematical discoveries. At 19, he figured out which regular polygons can be constructed using only a ruler and a compass. Gauss was so proud of this discovery that he asked for a regular heptadecagon (a polygon with 17 sides) to be carved on his tombstone.

Gauss did not limit himself to mathematics. He studied magnetism together with the famous physicist, Weber, and discovered a law that was named after him, “Gauss’s Law for Magnetism”. Gauss also studied optics and came up with a formula for lenses. His most ground-breaking contributions, however, were in the field of mathematics. Even we can understand statements of some of his results. For example, he proved that any whole number can be written as a sum of no more than three triangular numbers. Gauss died in 1855, but we can still encounter his name in many places. A “gauss” is the standard unit of measurement of magnetic influence (induction), named in his honor. There is a moon crater named Gauss, an asteroid christened Gaussia, an Antarctic volcano called Gaussberg, and so on. His portrait used to be on German banknotes before the Euros were introduced.



Leonardo Fibonacci (1170 – 1250)

Leonardo Fibonacci lived about 800 years ago. Nowadays he is best known for the “Fibonacci sequence”.

Fibonacci was born around 1170 in Pisa, Italy, into the family of a wealthy merchant, and died around 1250. Fibonacci travelled a lot with his father along the Mediterranean coast and saw how things were done in many European and North African countries. He probably spent much of his youth in the Algerian town of Bougie where his father held an official position.

At that time in Europe, everyone used Roman numerals: I, II, III, IV, V, VI, VII, VIII, IX, X, XI, . . . , and it was excruciatingly slow to do calculations using them. Meanwhile, the Arabs were using the same system as the one we use now, the Hindu-Arabic numbers.

When Fibonacci returned to Pisa, he decided to introduce Europeans to this better number system, and so he wrote the book “Liber Abaci”, which means “Book of the Calculators” (published in 1202). In one of the chapters of “Liber Abaci” Fibonacci described the sequence of numbers now called Fibonacci sequence. This book played an important role in spreading the Hindu-Arabic numeral system throughout Europe.

We only know a little about his life after this. He won a mathematics tournament in 1225 at the court of Pisa. His mathematical achievements were so valued that in 1240 he started to get paid a salary, which was almost unheard of in the Middle Ages!



Blaise Pascal (1623 – 1662)

“Renaissance men” are people who are extremely gifted and successful in many different areas. Blaise Pascal was definitely one of them.

He was a great inventor and a great writer. Even now, 350 years after he died, you can see people reading his books, “Letters” and “Thoughts”, in French buses (the bus service was invented by Pascal). He was also a philosopher, mathematician and physicist.

When Pascal was 10, he became interested in the sounds a china plate makes when tapped. Eventually he invented a series of physical experiments that showed that sound is the vibration of air.

Pascal was a sickly child. According to a family legend he was cursed by a witch. His father didn’t want to overburden the child, so he didn’t allow Pascal to study mathematics. However, the father was forced to change his mind. When Pascal was 12, he discovered on his own the most basic geometric fact about the sum of the angles of a triangle: the sum of the angles in any triangle is always equal to the straight angle, or 180 degrees. (This fact was discovered 2000 years earlier in Ancient Greece.)

Pascal became the first to develop geometry beyond what was known in Ancient Greece. Later he became famous for calculating odds of winning in card games. Among other things, Pascal studied behavior of liquids and air under pressure. The unit of pressure, *pascal*, is named after him.

Pascal constructed the first working mechanical calculator, called the Pascaline. Many other people before him had ideas about how to make mechanical calculators but never succeeded. The Pascaline was built using gears, and could add and subtract numbers. Many such calculators were built and sold. Nine of them still exist, and you can find them in museums in France.