Chapter 18

Which is Bigger? (a^b versus b^a)

Every now and then, I see problems that have been used in various mathematics contests, and every now and then, I am able to solve one of them, but almost never in the amount of time that would make me competitive. I am just not a mathematical gunslinger. There is one type of problem that I have seen more than once. It looks like this:

$$\text{Which is bigger, } 7.01^7 \text{ or } 7^{7.01}?$$

It was always pretty clear that using a calculator was not allowed. I had not permitted myself to be seduced by such problems until I saw the question in terms of two numbers that I really cared about:

$$\text{Which is bigger, } e^\pi \text{ or } \pi^e?$$

I went after this question and was able to show in a few different ways that e^π is the larger of the two. What began to gnaw at me is that none of the approaches seemed to be specific to π.

Here is the most useful approach for this discussion; the one that made it clear to me what was going on. Suppose we want to compare a^b with b^a. We will assume that a and b are both positive. The "log" used below is the natural log, logarithm base $e = 2.71828\ldots$. The question marks below indicate that we do not know which expression is bigger:

$$a^b \,?\, b^a,$$

$$b \log a \,?\, a \log b.$$

Hoping that I can discern an inequality that I can walk back to my original comparison, I multiply on both sides of the question mark by $1/ab$:

$$\frac{\log a}{a} \,?\, \frac{\log b}{b}.$$

This prods us to take a look at the function

$$f(x) = \frac{\log x}{x}.$$

We do this because we don't know what else to do. Desperation pays off, because I (and maybe you) have calculus in my tool kit. We would like to know where f achieves its maximal value, if it achieves a maximal value at all. Take the derivative and set it to zero to find out where the tangent line is horizontal!

$$f'(x) = \frac{1 - \log x}{x^2} = 0 \text{ only at } x = e.$$

Since we also manage to notice that the derivative is negative for $x > e$ and positive for $0 < x < e$, our function is maximized for the values of concern to us at $x = e$. Now we can slay the dragon. Our maximum occurs at e. So

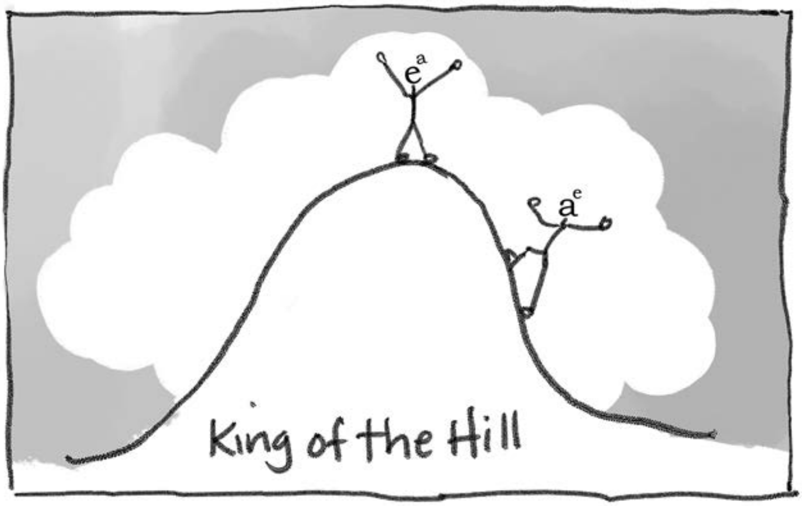
$$f(e) = \frac{\log e}{e} = \frac{1}{e} > \frac{\log a}{a}, \text{ for any } a > 0, a \neq e.$$

We backtrack.

$$a > e \log a = \log a^e.$$

Finally,

$$e^a > e^{\log a^e} = a^e, a \geq 0, a \neq e. \text{ (We can obviously include } a = 0.)$$



Choosing π was just a red herring. We can pick any non-negative number not equal to e. This fact characterizes e. But the method of proof tells us more. Since our function f is increasing for $x < e$ and decreasing for $x > e$, we see that

$$0 \leq a < b < e \Rightarrow a^b < b^a,$$
$$e < a < b \Rightarrow a^b > b^a.$$

In general we cannot make a determination immediately, if one of our numbers is less than e and the other is greater.

We can pull one more nugget from this discussion. If a and b are distinct integers and $a^b = b^a$, then the pair a and b must be 2 and 4. We see this in the following way. For any integer $n > 4$

$$\frac{\log 4}{4} > \frac{\log n}{n},$$

because f is decreasing for $x > e = 2.71828\ldots$. It follows that

$$\frac{\log 2}{2} = \frac{\log 4}{4} > \frac{\log n}{n},$$

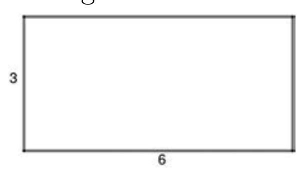
so $2^n > n^2$ if $n > 4$ and, since $2^3 < 3^2$, we have completed the case for when one of our integers is 2.

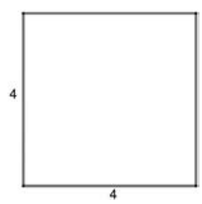
Clearly, if we pick two integers greater than e, they cannot satisfy $a^b = b^a$, again because of the decreasing nature of f. That leaves us with 2 and 4 being the unique pair that satisfies $a^b = b^a$.

At any rate, it is absolutely clear now that

$$7^{7.01} > 7.01^7.$$

Laying the groundwork to show that 2 and 4 are the only integers that satisfy $a^b = b^a$ was not what I would call easy. It reminds me of another question. How many ways can a rectangle be constructed so that its area equals its perimeter, if the length of each edge must be a whole number? Here are two examples of rectangles that meet the requirements.





Are there others? Initially one might think that this is as difficult to address as our $a^b = b^a$ inquiry. However, that is not the case. Our rectangles with dimensions $a \times b$ must satisfy

$$ab = 2a + 2b.$$

Following our nose, we solve for a and get

$$a = \frac{2b}{b-2} = 2 + \frac{4}{b-2}.$$

Then we remember that a and b must be positive integers. Right off the bat that forces $b - 2$ to be a divisor of 4. After eliminating $b = 1$, 2, and 5, we see that the only positive integers that do the job as choices for b are 3, 4, and 6, corresponding to 6, 4, and 3 for a, respectively. If b is bigger than 6, $b - 2$ is not a divisor of 4.

Cute and not hard. You can never be sure what the nature of the challenge will be when you open your mouth and pose a question.