

## Essay 1

# Dragons and Poison

Here's a classic puzzle. I am not sure of its origin, but I have seen the puzzle also presented as a "six well puzzle". I was first told this puzzle by mathematics educator Avery Pickford.

### **Dragons and Poison**

You and a dragon have agreed to take part in the following "game". (I am not sure why but, well, that is how it is). At noon today you will each bring to the local coffee house a goblet of poison. The dragon will take a sip of poison from your goblet and then a sip from his own. You will take a sip of poison from the dragon's goblet and then a sip from your own. You will then each sit and wait for the results.

Let me tell you about the poison.

There is only one type of poison available to each of you and it comes in varying strengths of potency. A single sip of any potency is enough for quite a detrimental effect (namely your or the dragon's complete demise), but it will take a few hours to act. There is an antidote to the poison: a sip of a stronger dose of the poison. Taking two sips—one dose followed by a stronger dose—has the same effect as not taking any poison at all. However, taking a second dose of equal or weaker potency will not help your predicament one whit.

Let me tell you something about the dragon.

She has access to the most potent strength of poison of all. (And you don't!)

So here is the challenge.

Given this knowledge of how poison works and the fact that the dragon might bring the most potent sample of all, is there a means for you to survive this cheery game?

### 1.1. Analyzing the Puzzle

Here is a first answer to our conundrum.

The dragon has nothing to gain by bringing anything but the strongest poison available. She will sip your poison and then be cured from it by sipping her own. You, on the other hand, will sip the strongest poison first and will not be cured by a sip of your own since it is sure to be weaker. Your doom is guaranteed.

**Dragon:** Brings strongest poison  
**You:** Bring weaker poison  
**Who survives:** Dragon

But you can be clever! You know that the dragon will reason to bring the strongest poison, so use it as a cure for yourself. Simply sip some poison just before the game and bring a goblet of water to the coffee house. You will be cured by the dragon's poison, and the dragon will be killed by her own poison since you brought only water.

**Dragon:** Brings strongest poison  
**You:** Bring water and sip poison beforehand  
**Who survives:** You

This sounds grand, but let's not underestimate the intelligence of dragons! She knows that you will reason she will bring the strongest poison and that you will operate in accordance—bringing water to the coffee house and drinking a dose of poison just before the game. She can foil your move by actually bringing water to the coffee house. You will be doomed and she will survive. (As another option, the dragon could, instead, bring the strongest poison to the table and sip some weak poison beforehand like you. This will lead to survival for each of you. But witnessing your demise seems more dragon-like. She will opt for that.)

**Dragon:** Brings water  
**You:** Bring water and sip poison beforehand  
**Who survives:** Dragon

However, you too are no fool and will reason that the dragon will reason that this is how you will reason about how she reasons! You can foil this dragon's change of strategy by not drinking anything at all beforehand (and you will both drink water and both survive) or, better yet, you drink a mild dose of poison before the game and bring along a stronger dose. You will survive and the dragon will not.

**Dragon:** Brings water  
**You:** Bring poison and sip weaker poison beforehand  
**Who survives:** You

But the dragon knows you will operate this way and so will reason that you will reason that she shall reason that . . . .

This game actually has no stable solution if both players are rational thinkers: For any set of choices each player may make (bring poison or

bring water, sip poison beforehand or not) there is reason for at least one of the players to change strategy. I don't have any particular advice for you if you are ever invited to play such a game with a dragon!

**Experiment.** It would be interesting to test human psychology and play a nonlethal version of the game. Cards from a deck can represent doses of poison. Each of two players can select a card, two through king, bring it to the table, and perhaps slip a second card in her pocket to represent a dose of poison ingested beforehand. The person playing the dragon has the option to bring an ace card, trumping all other values.

Have a class of students play the game. What seems to be the most common human strategy for the human player? For the dragon? How do strategies change if players play against each other multiple times and learn about each other's thinking?

### Research Corner

Given the lack of a stable solution to the game and the inherent randomness in what each player will settle to do, game theorists would advise you and the dragon each employ a "mixed strategy" and maximize any advantage randomness could offer. You each have two choices to make:

- (1) bring poison or bring water,
- (2) sip weak poison beforehand or not.

(You might argue it behooves the dragon to bring the strongest dose of poison if she brings any at all.)

A mixed strategy has you each flip a coin to decide what to do, but make it a biased coin. Let  $d_1$  be the probability that the dragon will choose to bring poison, and  $d_2$  the probability she will sip poison beforehand. Let  $y_1$  and  $y_2$  represent your two corresponding probabilities. Are there values for  $d_1$ ,  $d_2$ ,  $y_1$ ,  $y_2$  that simultaneously maximize each of your chances of survival?

Is this problem more manageable if there are only three levels of poison and water available? You have access to the two weaker poisons and the dragon has access to all three.



## Essay 2

# Folding Tetrahedra

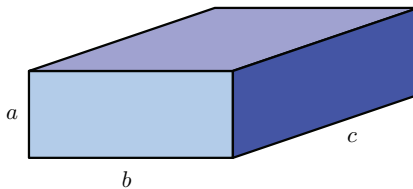
Two puzzles:

A rectangular box has six rectangular faces, each of the same area. Must that box be a cube?

A tetrahedral box has four triangular faces, each of the same area. Must that box be a regular tetrahedron (that is, have four congruent equilateral triangle faces)?

### 2.1. Polyhedron Symmetry

Consider a rectangular box with side lengths  $a$ ,  $b$ , and  $c$ .



If each face has the same area, then we have  $ab = bc = ac$ . The first equality gives  $a = c$  and the second  $a = b$ . Thus, all three side-lengths are the same and the figure is indeed a cube.

One might suspect that a tetrahedron with faces of the same area must be regular too, but this is not actually the case! Look at the tetrahedron formed by the diagonals of a noncube box.

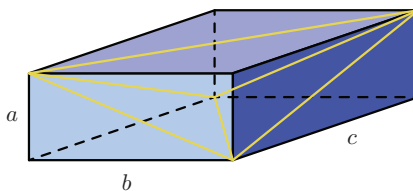
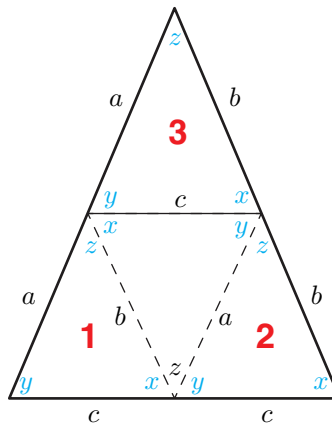


FIGURE 2.1

Each triangular face has side lengths  $\sqrt{a^2 + b^2}$ ,  $\sqrt{b^2 + c^2}$ , and  $\sqrt{a^2 + c^2}$ , and thus all faces are congruent and so have the same area, but the faces are not equilateral if  $a$ ,  $b$ , and  $c$  are not all equal.

**Challenge.** Find another tetrahedron with congruent faces situated in a rectangular box by connecting the midpoints of some edges of the box.

One can obtain other examples of such tetrahedra by cutting out a paper triangle with three acute angles. Connecting midpoints of its sides with line segments divides the triangle into four congruent subtriangles.



Lay the paper triangle on a table top and fold the paper along the two dotted lines to tilt subtriangles 1 and 2 into three-dimensional space. Since angle  $z$  is smaller than  $90^\circ$ , we have that  $x + y > 90^\circ$ , and so these two triangles can't fold back flat onto the table top without overlap. So there is an intermediate position in three-dimensional space where these two triangles meet along their edges of common length  $c$ . Now fold along the third midpoint line to tilt triangle 1 into three-dimensional space. Its side-lengths  $a$  and  $b$  align perfectly with the space formed by triangles 1 and 2 to allow us to make a tetrahedron.

Thus from any acute triangle we can construct a nonregular tetrahedron with four congruent faces necessarily of the same area.

**Challenge.** Figure 2.1 shows how to find a tetrahedron with congruent faces sitting inside any given rectangular box. Are the faces of such tetrahedra necessarily acute triangles? Can every tetrahedron with congruent acute triangular faces be situated in a rectangular box in this way?

**Research Corner**

Is there an example of a tetrahedron with four faces of the same area with at least one face an obtuse triangle?

Find a general formula for the volume of a tetrahedron with four congruent acute-triangle faces.

If a rectangular box has six faces of the same perimeter, it must be a cube. (Why?) What can you say about a tetrahedron with four triangular faces of the same perimeter?

The Wikipedia page on disphenoids, <https://en.wikipedia.org/wiki/Disphenoid>, makes a number of interesting claims about the tetrahedra discussed in this essay. Are the mathematical claims true?