

Contents

Preface	xi
Topics Explored	xiii
Essay 1. Dragons and Poison	1
1.1. Analyzing the Puzzle	2
Essay 2. Folding Tetrahedra	5
2.1. Polyhedron Symmetry	5
Essay 3. The Arbelos	9
3.1. The Arbelos	10
3.2. The Area of the Arbelos	10
3.3. The Archimedean Circles	11
3.4. The Other Common Tangent Segment	12
3.5. Conic Curves	13
Essay 4. Averages via Distances	17
4.1. Ideal vs. Real Data	17
4.2. Using the Distance Formula: Euclidean Distance	17
4.3. Using the Taxicab Metric	18
4.4. Mean, Median, and . . .	20
Essay 5. Ramsey Theory	23
5.1. Ramsey Theory	23
5.2. Connections to the Opening Puzzler	27
Essay 6. Inner Triangles	29
6.1. Two Observations about Triangles	30
6.2. Routh's Theorem	31
6.3. Solving Feynman's Problem without the Big Guns	33
Essay 7. Land or Water?	37
7.1. The Answer to the First Puzzler	39
7.2. Towards Answering the Second Puzzler	40
7.3. Proving the Theorem	41

Essay 8. Escape	45
8.1. Leibniz's Harmonic Triangle	46
8.2. The Infinite Stocking Property	47
8.3. Variations of Leibniz's Harmonic Triangle	48
8.4. Solving the Opening Puzzle	49
Essay 9. Flipping a Coin for a Year	53
9.1. Solutions to Coin Tossing 1	54
9.2. Solutions to Coin Tossing 2	56
Essay 10. Coinciding Digits	59
10.1. The Chinese Remainder Theorem	59
10.2. The Opening Puzzler	61
Essay 11. Inequalities	63
11.1. Solving Puzzle 1	64
Essay 12. Gauss's Shoelace Formula	67
12.1. Step 1: Nicely Situated Triangles	71
12.2. Step 2: General Triangles	73
12.3. Step 3: Begin Clear of the Effect of Motion	74
12.4. Step 4: Being Clear on Starting Points	75
12.5. Step 5: Steps 1 and 2 Were Unnecessary!	76
12.6. Step 6: Quadrilaterals	76
12.7. Step 7: Beyond Quadrilaterals	78
Essay 13. Subdividing a Square into Triangles	81
13.1. Sperner's Lemma	82
13.2. The Impossibility Proof	83
13.3. Case 1: N is an Odd Integer	84
13.4. Case 2: N is an Even Integer	85
Essay 14. Equilateral Lattice Polygons	89
14.1. Areas of Lattice Polygons	90
14.2. Aside: Pick's Theorem	91
14.3. Equilateral Lattice Polygons	92
14.4. The Answer with Cheating	93
14.5. Dots of Zero Width	94
Essay 15. Broken Sticks and Viviani's Theorem	99
15.1. Viviani's Theorem	100
15.2. Broken Sticks and Triangles	101
15.3. Something Unsettling	104
15.4. Focus on the Left Piece	105
15.5. Summing Probabilities	107
15.6. Answers	108

Essay 16. Viviani's Converse?	109
16.1. Planes above Triangles	109
16.2. The Equation of a Plane	110
16.3. The Distance of a Point from a Line	111
16.4. The Converse of Viviani's Theorem	114
16.5. Other Figures	115
Essay 17. Integer Right Triangles	117
17.1. A Cute Way to Find Pythagorean Triples	118
17.2. A Primitive Tidbit	120
17.3. The Answers to All the Curiosities	120
Essay 18. One More Question about Integer Right Triangles	127
18.1. A Precursor Question	127
18.2. The Answer to the Main Question	128
Essay 19. Intersecting Circles	131
19.1. Loops on a Page	132
19.2. Circles on a Page	138
19.3. Polygons on a Page	139
Essay 20. Counting Triangular and Square Numbers	141
20.1. Some Interplay between Square and Triangular Numbers	142
20.2. Formulas	143
20.3. Counting Figurate Numbers	144
20.4. The Squangular Numbers	146
20.5. Something Bizarre!	147
Essay 21. Balanced Sums	149
21.1. On Sums of Consecutive Counting Numbers	149
21.2. How to Find More!	154
21.3. To Summarize	156
Essay 22. The Prouhet–Thue–Morse Sequence	159
22.1. The Prouhet–Thue–Morse Sequence	159
22.2. The Opening Puzzler	160
22.3. Alternative Constructions	162
22.4. Proving the Puzzler	165
Essay 23. Some Partition Numbers	169
23.1. The Partition Numbers	169
23.2. Partitions into a Fixed Number of Parts	170
23.3. Cracking the $P_3(n)$ Formula	171
Essay 24. Ordering Colored Fractions	177
24.1. Coloring and Ordering Fractions	177
24.2. A Side Track	179
24.3. The Mediant	180

24.4.	Explaining Colored Fractions	181
24.5.	A Bad Example?	185
24.6.	Addendum	186
Essay 25.	How Round Is a Cube?	187
25.1.	Deficiencies in Surface Circles	188
25.2.	Rounding the Cube	190
25.3.	A Better Way to Count Total Pointiness	192
25.4.	Shaving Corners Does Not Help!	196
25.5.	Not All Shapes Are Sphere-like!	197
25.6.	Christopher Columbus and Others	198
Essay 26.	Base and Exponent Switch	201
26.1.	The Graph of $y = x^{\frac{1}{x}}$ for $x > 0$	201
26.2.	The Graph of $x^y = y^x$ for $x > 0, y > 0$	202
26.3.	A Connection to $w^{w^{w^{\dots}}}$	203
26.4.	Appendix: A Tricky Swift Proof	205
Essay 27.	Associativity and Commutativity Puzzlers	207
27.1.	Order	208
27.2.	Explaining the Puzzler and Its Variations	210
Essay 28.	Very Triangular and Very Very Triangular Numbers	215
28.1.	Numbers in Binary	215
28.2.	Very Triangular Numbers	217
Essay 29.	Torus Circles	219
29.1.	The Equation of a Torus	220
29.2.	Slicing an Ideal Bagel: The Puzzler	222
29.3.	The Four Circles Property	228
Essay 30.	Trapezoidal Numbers	231
30.1.	Trapezoidal Numbers	231
30.2.	From Rectangles to Trapezoids	233
30.3.	Two Transformations	234
30.4.	Counting Presentations	236
Essay 31.	Square Permutations	239
31.1.	Square Permutations	239
31.2.	Taking Stock	242
Essay 32.	Tupper's Formula	245
32.1.	Understanding the Notation	245
32.2.	The mod Function in Computer Science	246
32.3.	Encoding a Picture	247
32.4.	Delightful Recursive Quirkiness	249
32.5.	How to Find a Particular Picture	250

32.6. Tupper's Dimensions	250
Essay 33. Compositional Square Roots	251
33.1. Compositional Square Roots	251
33.2. Constructing Compositional Square Roots	253
Essay 34. Polynomial Permutations	257
34.1. Playing with Polynomials	258