Foreword

Problems, exercises, circles, and olympiads

This is a translation of Part 1 of the book *Mathematics Through Problems: From Mathematical Circles and Olympiads to the Profession*, and is part of the MSRI Mathematical Circles Library series. The other two parts, *Geometry* and *Combinatorics*, will be published in the same series soon.

The goal of the MSRI Mathematical Circles Library series is to build a body of works in English that help to spread the "math circle" culture. A *mathematical circle* is an eastern-European notion. Math circles are similar to what most Americans would call a math club for kids, but with several important distinguishing features.

First, they are *vertically integrated*: young students may interact with older students, college students, graduate students, industrial mathematicians, professors, and even world-class researchers, all in the same room. The circle is not so much a classroom as a gathering of young initiates with elder tribespeople, who pass down *folklore*.

Second, the "curriculum," such as it is, is dominated by *problems* rather than specific mathematical topics. A problem, in contrast to an exercise, is a mathematical question that one doesn't know how, at least initially, to approach. For example, "What is 3 times 5?" is an exercise for most people but a problem for a very young child. Computing 5^{34} is also an exercise, conceptually very much like the first example, certainly harder, but only in a "technical" sense. And a question like "Evaluate $\int_2^7 e^{5x} \sin 3x\, dx$" is also an exercise for calculus students—a matter of "merely" knowing the right algorithm and how to apply it.

Problems, by contrast, do not come with algorithms attached. By their very nature, they require *investigation*, which is both an art and a science, demanding technical skill along with focus, tenacity, and inventiveness. Math circles teach students these skills, not with formal instruction, but by having them *do math* and observe others doing math. Students learn that a problem worth solving may require not minutes but possibly hours, days, or even years of effort. They work on some of the classic folklore problems and discover how these problems can help them investigate other problems. They learn how not to give up and how to turn errors or failures into opportunities for more investigation. A child in a math circle learns to do exactly what a research mathematician does; indeed, he or she does independent research, albeit on a lower level, and often—although not always—on problems that others have already solved.

Finally, many math circles have a culture similar to a sports team, with intense camaraderie, respect for the "coach," and healthy competitiveness (managed wisely, ideally, by the leader/facilitator). The math circle culture is often complemented by a variety of problem solving contests, often called *olympiads*. A mathematical olympiad problem is, first of all, a genuine problem (at least for the contestant), and usually requires an answer which is, ideally, a well-written argument (a "proof").

Why this book, and how to use it

The Mathematical Circles Library editorial board chose to translate this book because it has an audacious goal—promised by its title—to develop mathematics through problems. This is not an original idea, nor just a Russian one. American universities have experimented for years with IBL (inquiry-based learning) and Moore-method courses, structured methods for teaching advanced mathematics through open-ended problem solving.[1]

But this massive work is an attempt to curate sequences of problems for secondary students (the stated focus is high school students, but that can be broadly interpreted) that allow them to discover and recreate much of "elementary" mathematics (number theory, polynomials, inequalities, calculus, geometry, combinatorics, game theory, probability) and start edging into the sophisticated world of group theory, Galois theory, etc.

The book is not possible to read from cover to cover—nor should it be. Instead, the reader is invited to start working on problems that he or she finds appealing and challenging. Many of the problems have hints and solution sketches, but not all. No reader will solve all the problems. That's not the point—it is not a contest. Furthermore, some of the problems are not supposed to be solved, but should rather be pondered. For example, when learning about primes, it is natural to wonder whether there is always a prime between n and $2n$. Indeed, this is problem 1.6.9 (c)—the very nontrivial result known as Bertrand's postulate—and the text provides references for learning more about it. Just because it is "too advanced" doesn't mean that it shouldn't be thought about! In fact, sometimes the reader is explicitly directed to jump ahead, with references to material that appears later in the book (the authors assure the reader that this will not lead to circular reasoning).

Indeed, this is the philosophy of the book: Mathematics is not a sequential discipline, where one is presented with a definition that leads to a lemma which leads to a theorem which leads to a proof. Instead it is an adventure

[1] See, for example, `https://en.wikipedia.org/wiki/Moore_method` and `http://www.jiblm.org`.

filled with exciting side trips as well as wild goose chases. The adventure is its own reward, but it also, fortuitously, leads to deep understanding and appreciation of mathematical ideas that cannot be accomplished by passive reading.

English-language references

Most of the references cited in this book are in Russian. However, there are many excellent books in English (some translated from Russian). Here is a very brief list, organized by topic and chapter.[2]

Articles from *Kvant*: This superb journal is published in Russian. However, it has been sporadically translated into English under the name *Quantum*, and there are several excellent collections in English; see [**FT07**, **Tab99**, **Tab01**].

Problem collections: *The USSR Olympiad Problem Book* [**SC93**] is a classic collection of carefully discussed problems. Additionally, [**FK91**] and [**FBKY11a**, **FBKY11b**] are good collections of olympiads from Leningrad and Moscow, respectively. See also the nicely curated collections of fairly elementary Hungarian contest problems [**Kur63a**, **Kur63b**, **Liu01**] and the more advanced (undergraduate-level) Putnam Exam problems [**KPV02**].

Inequalities: See [**Ste04**] for a comprehensive guide and [**AS16b**] for a more elementary text. The author also recommends two classic books, [**HLP67**] and [**BB65**], and the more specialized text [**MO09**], but cautions that these are all rather advanced.

Geometry: *Geometry Revisited* [**CG67**] is a classic, and [**Che16**] is a recent and very comprehensive guide to "olympiad geometry."

Polynomials and theory of equations: See [**Bar03**] for an elementary guide, and [**Bew06**] for a historically motivated exposition of constructability and solvability and unsolvability. In Chapter 8, see the book [**Gin07**] for English translations of the *Kvant* articles [**Gin72**, **Gin76**], and [**Skoa**] for an abridged English version of [**Sko10**].

Combinatorics: The best book in English, and possibly any language, is *Concrete Mathematics* [**GKP94**].

Functions, limits, complex numbers, and calculus: The classic *Problems and Theorems in Analysis* by Pólya and Szegő [**PS04**] is—like the current text—a curated selection of problems but at a much higher mathematical level.

Paul Zeitz
April 2019

[2] We omit any supplementary Russian-language references mentioned in the original text that were not actually cited in the text.

Introduction

What this book is about and who it is for

A deep understanding of mathematics is useful both for mathematicians and for high-tech professionals. In particular, the "profession" in the title of this book does not necessarily mean the profession of mathematics.

This book is intended for high school students and undergraduates (in particular, those interested in olympiads). For more details, see "Olympiads and mathematics" on p. xvii. The book can be used both for self-study and for teaching.

This book attempts to build a bridge (by showing that there is no gap) between ordinary high school exercises and the more sophisticated, intricate, and abstract concepts in mathematics. The focus is on engaging a wide audience of students to think creatively in applying techniques and strategies to problems motivated by "real world or real work" [**Mey**]. Students are encouraged to express their ideas, conjectures, and conclusions in writing. Our goal is to help students develop a host of new mathematical tools and strategies that will be useful beyond the classroom and in a number of disciplines (cf. [**IBL**, **Mey**, **RMP**]).

The book contains the most standard "base" material (although we expect that at least some of this material is review—that not all is being learned for the first time). But the main content of the book is more complex material. Some topics are not well known in the traditions of mathematical circles, but are useful both for mathematical education and for preparation for olympiads.

The book is based on the classes taught by the author at different times at the Independent University of Moscow, at various Moscow schools and math circles, in preparing the Russian team for the International Mathematical Olympiad, in the "Modern Mathematics" summer school, in the Kirov and Kostroma Summer Mathematical Schools, in the Moscow Olympiad School, and also in the summer Conference of the Tournament of Towns.

Much of this book is accessible to high school students with a strong interest in mathematics.[3] We provide definitions or references for material that is not standard in the school curriculum. However, many topics are difficult if you study them "from scratch." Thus, the ordering of the problems helps to provide "scaffolding." At the same time, many topics are *independent* of each other. For more details, see p. xviii, "How this book is organized".

Learning by doing problems

We subscribe to the tradition of studying mathematics by solving and discussing problems. These problems are selected so that in the process of solving them the reader (more precisely, the solver) masters the fundamentals of important ideas, both classical and modern. The main ideas are developed incrementally with olympiad-style examples—in other words, by the simplest special cases, free from technical details. In this way, we show *how you can explore and discover these ideas on your own.*

Learning by solving problems is not just a serious approach to mathematics; it also continues a venerable cultural tradition. For example, the novices in Zen monasteries study by reflecting on riddles ("koans") given to them by their mentors. (However, these riddles are rather more like paradoxes than what we consider to be problems.) See, for example, [**Suz18**]; compare with [**Pla12**, pp. 26–33]. "Math" examples include [**Arn16b**, **BSe**, **RSG+16**, **KBK08**, **Pra07b**, **PS04**, **SC93**, **Sko09**, **Vas87**, **Zvo12**], which sometimes describe not only problems but also the principles of selecting appropriate problems. For the American tradition, see [**IBL**, **Mey**, **RMP**].

Learning by solving problems is difficult, in part, because it generally does not create the *illusion* of understanding. However, one's efforts are fully rewarded by a deep understanding of the material, leading to the ability to carry out similar (and sometimes rather different) reasoning. Eventually, while working on fascinating problems, readers will be following the thought processes of the great mathematicians and may see how important concepts and theories naturally evolve. Hopefully this will help them make their own equally useful discoveries (not necessarily in math)!

Solving a problem, theoretically, requires only understanding its statement. Other facts and concepts are not needed. (Actually, useful facts and ideas will be developed while solving the problems presented in this book.) Sometimes, you may need to know things from other parts of the book as indicated in the instructions and hints. For the most important problems we provide hints, instructions, solutions, and answers, located at the end of

[3] Some of the material is studied in math circles and summer schools by those who are just getting acquainted with mathematics (for example, 6th graders). However, the presentation here is aimed at the reader who already has at least a minimal understanding of mathematical culture. Younger students need a different approach; see, for example, [**GIF94**].

each section. However, they should be referred to only after attempting to solve a problem.

As a rule, we present the *formulation* of a beautiful or important result (in the form of a problem) before its *proof.* In such cases, one may need to solve later problems in order to fully work out the proof. This is always explicitly mentioned in hints, and sometimes even in the text. Consequently, if you fail to solve a problem, please read on. This guideline is helpful because it simulates the typical research situation.

This book "is an attempt to demonstrate learning as *dialogue* based on solving and discussing problems" (see [**KBK15**]).

A message ***By A. Ya. Kanel-Belov***

To solve difficult olympiad problems, at the very least one must have a robust knowledge of algebra (particularly algebraic transformations) and geometry. Most olympiad problems (except for the easiest ones) require "mixed" approaches; rarely is a problem resolved by applying a method or idea in its pure form. Approaching such mixed problems involves combining several "crux" problems, each of which may involve single ideas in a "pure" form. *Learning to manipulate algebraic expressions is essential. The lack of this skill among olympians often leads to ridiculous and annoying mistakes.*

Olympiads and mathematics

To him a thinking man's job was not to deny one reality
at the expense of the other, but to include and to connect.

U. K. Le Guin. *The Dispossessed.*

Here are three common misconceptions about very worthwhile goals: the best way to prepare for a math olympiad is by solving last year's problems; the best way to learn "serious" mathematics is by reading university textbooks; the best way to master any other skill is with no math at all. There is a further misconception that one cannot achieve these apparently divergent goals simultaneously. The authors share the widespread belief that these three approaches miss the point and lead to harmful side effects: students become too keen on emulation, or they study the *language* of mathematics rather than its *substance*, or they underestimate the value of robust math knowledge in other disciplines.

We believe that these three goals are not as divergent as they might seem. The foundation of mathematical education should be the *solution and discussion of problems interesting to the student, during which a student learns important mathematical facts and concepts.* This simultaneously prepares the student for math olympiads and the "serious" study of mathematics, and is good for his or her general development. Moreover, it is more effective for achieving success in any one of the three goals above.

Research problems for high school students

Many talented high school or university students are interested in solving research problems. Such problems are usually formulated as complex questions broken into incremental steps; see, e.g., [**LKT**]. The final result may even be unknown initially, appearing naturally only in the course of thinking about the problem. Working on such questions is useful in itself and is a good approximation to scientific research. Therefore it is useful if a teacher or a book can support and develop this interest.

For a description of successful examples of this activity, see, for example, projects in the Moscow Mathematical Conference of High School Students [**M**]. While most of these projects are not completely original, sometimes they can lead to new results.

How this book is organized

You should not read each page in this book, one after the other. You can choose a sequence of study that is convenient for you (or omit some topics altogether). Any section (or subsection) of the book can be used for a math circle session.

The book is divided into chapters and sections (some sections are divided into subsections), with a plan of organization outlined at the start of each section. If the material of another section is needed in a problem, you can either ignore it or look up the reference. This allows greater freedom when studying the book, but at the same time it may require careful attention.

Topics of each section are arranged approximately in order of increasing complexity. The numbers in parentheses after a topic name indicate its "relative level": 1 is the simplest, and 4 is the most difficult. The first topics (not marked with an asterisk) are basic; unless indicated otherwise, you should begin your study with them. The remaining ones (marked with an asterisk) can be returned to later; unless otherwise stated, they are independent of each other. As you read, try to *return* to old material, but at a new level. Thus you should end up studying different levels of a topic *not sequentially* but as part of a mixture of topics.

The notation used throughout the book is given on p. xx. Notation and conventions particular to a specific section are introduced at the beginning of that section. The book concludes with a subject index. The numbers in bold are the pages on which *formal definitions* of concepts are given.

Sources and literature

Each chapter ends with a bibliography that relates to the entire chapter, with sources for each topic.[4] For example, we cite the book [**GKP94**],

[4] *Editor's note:* In the English edition all the references are combined into one list at the end of the book.

which involves both combinatorics and algebra. In addition to sources for specialized material, we also tried to include the very best popular writing on the topics studied. We hope that this bibliography, at least as a first approximation, can guide readers through the sea of popular scientific literature in mathematics. However, the great size of this genre guarantees that many remarkable works had to be omitted. Please note that items in the bibliography are not necessary for solving the problems in this book, unless explicitly stated otherwise.

Many of the problems are not original, but the source (even if it is known) is usually not specified. When a reference is provided, it comes after the statement of the problem, so that the reader can compare his or her solution with the one given there. When we know that many problems in a section come from one source, we mention this.

We do not provide links to online versions of articles in the popular magazines *Kvant* (the English magazine *Quantum* is based on *Kvant*) and *Matematicheskoe Prosveshchenie* ("Mathematical Enlightment"); they can be found at the websites `http://kvant.ras.ru`, `http://kvant.mccme.ru`, `https://en.wikipedia.org?wiki?Quantum_Magazine`, and `http://www.mccme.ru/free-books/matpros.html`.

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We apologize for any mistakes, and will be grateful to readers for pointing them out.

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Numbering and notation

Sections in each chapter are arranged approximately in order of increasing complexity of the material. The numbers in parentheses after the section name indicate its "relative level": 1 is the simplest, and 4 is the most difficult. The first sections (not marked with an asterisk) are basic; unless indicated otherwise, you can begin to study the chapter with them. The remaining sections (marked with an asterisk) can be returned to later; unless otherwise stated, they are independent of each other.

If a mathematical fact is formulated as a problem, then the objective is to prove this fact. Open-ended questions are called *challenges*; here one must come up with clear wording and a proof; cf., for example, [**VINK10**].

The most difficult problems are marked with asterisks (*). If the statement of the problem asks you to "find" something, then you need to give a "closed form" answer (as opposed to, say, an unevaluated sum of many terms).

Once again, if you are unable to solve a problem, continue reading: later problems may turn out to be hints.

Notation

• $\lfloor x\rfloor = [x]$ — (lower) integer part of number x ("floor"); that is, the largest integer not exceeding x.
• $\lceil x\rceil$ — upper integer part of number x ("ceiling"); that is, the smallest integer not less than x.
• $\{x\}$ — fractional part of number x; equal to $x - \lfloor x\rfloor$.
• $d|n$, or $n \,\vdots\, d$ — d *divides* n; that is, there exists an integer k such that $n = kd$ (the number d is called a *divisor* of the number n; we assume that $d \neq 0$).
• $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ — the sets of all real numbers, rational numbers, and integers, respectively.
• $\mathbb{Z}_2$ — the set $\{0, 1\}$ of remainders upon division by 2 with the operations of addition and multiplication modulo 2.
• $\mathbb{Z}_m$ — the set $\{0, 1, \ldots, m-1\}$ of remainders upon division by m with the operations of addition and multiplication modulo m. (Specialists in algebra often write this set as $\mathbb{Z}/m\mathbb{Z}$ and use $\mathbb{Z}_m$ for the set of *m-adic integers* for the prime m.)
• $\binom{n}{k}$ — the number of k-element subsets of an n-element set (also denoted by C_n^k).
• $|X|$ — the number of elements in set X.
• $A - B = \{x \mid x \in A \text{ and } x \notin B\}$ — the difference of the sets A and B.
• $A \sqcup B$ — the disjoint union of the sets A and B; that is, the union of A and B viewed as the union of disjoint sets.

• $A \subset B$ — means the set A is contained in the set B. In some books, this is denoted by $A \subseteq B$, and $A \subset B$ means "the set A is in the set B and is not equal to B."
• We abbreviate the phrase "Define x to be a" to $x := a$.