

Contents

Foreword	xi
Problems, exercises, circles, and olympiads	xi
Why this book, and how to use it	xii
English-language references	xiii
Introduction	xv
What this book is about and who it is for	xv
Learning by doing problems	xvi
A message <i>By A. Ya. Kanel-Belov</i>	xvii
Olympiads and mathematics	xvii
Research problems for high school students	xviii
How this book is organized	xviii
Sources and literature	xviii
Acknowledgments	xix
Grant support	xix
Numbering and notation	xx
Notation	xx
Chapter 1. Divisibility	1
1. Divisibility (1)	1
Suggestions, solutions, and answers	2
2. Prime numbers (1)	4
Suggestions, solutions, and answers	5
3. Greatest common divisor (GCD) and least common multiple (LCM) (1)	6
Suggestions, solutions, and answers	7
4. Division with remainder and congruences (1)	8
Hints	9
5. Linear Diophantine equations (2)	10
Suggestions, solutions, and answers	11
6. Canonical decomposition (2*)	12
Suggestions, solutions, and answers	14
7. Integer points under a line (2*)	14
Suggestions, solutions, and answers	15

Chapter 2. Multiplication modulo p	17
1. Fermat's Little Theorem (2)	17
Suggestions, solutions, and answers	18
2. Primality tests (3*) <i>By S. V. Konyagin</i>	19
Hints	20
Suggestions, solutions, and answers	20
3. Quadratic residues (2*)	21
Hints	22
Suggestions, solutions, and answers	22
4. The law of quadratic reciprocity (3*)	23
Suggestions, solutions, and answers	24
5. Primitive roots (3*)	26
Suggestions, solutions, and answers	27
6. Higher degrees (3*) <i>By A. Ya. Kanel-Belov and A. B. Skopenkov</i>	28
Hints	29
Suggestions, solutions, and answers	29
Chapter 3. Polynomials and complex numbers	31
1. Rational and irrational numbers (1)	31
Suggestions, solutions, and answers	32
2. Solving polynomial equations of the third and fourth degrees (2)	34
Hints	35
Suggestions, solutions, and answers	36
3. Bezout's Theorem and its corollaries (2)	38
Suggestions, solutions, and answers	40
4. Divisibility of polynomials (3*) <i>By A. Ya. Kanel-Belov and A. B. Skopenkov</i>	41
Hints and answers	42
5. Applications of complex numbers (3*)	43
Hints and answers	45
6. Vieta's Theorem and symmetric polynomials (3*)	46
Suggestions, solutions, and answers	47
7. Diophantine equations and Gaussian integers (4*) <i>By A. Ya. Kanel-Belov</i>	47
Suggestions, solutions, and answers	49
8. Diagonals of regular polygons (4*) <i>By I. N. Shnurnikov</i>	51
Suggestions, solutions, and answers	52
9. A short refutation of Borsuk's conjecture	53
Suggestions, solutions, and answers	56
Chapter 4. Permutations	59
1. Order, type, and conjugacy (1)	59
Hints and answers	62

2. The parity of a permutation (1)	62
Hints and answers	63
3. The combinatorics of equivalence classes (2)	64
Answers	68
Chapter 5. Inequalities	69
1. Towards Jensen's inequality (2)	69
Hints	71
Suggestions, solutions, and answers	72
2. Some basic inequalities (2)	73
Hints	75
Suggestions, solutions, and answers	75
3. Applications of basic inequalities (3*) <i>By M. A. Bershtein</i>	75
Hints	77
Suggestions, solutions, and answers	78
4. Geometric interpretation (3*)	82
Suggestions, solutions, and answers	83
Chapter 6. Sequences and limits	85
1. Finite sums and differences (3)	85
Hints	86
Suggestions, solutions, and answers	87
2. Linear recurrences (3)	88
Hints	89
Suggestions, solutions, and answers	90
3. Concrete theory of limits (4*)	90
Suggestions, solutions, and answers	92
4. How does a computer calculate the square root? (4*) <i>By</i> <i>A. C. Vorontsov and A. I. Sgibnev</i>	93
Suggestions, solutions, and answers	94
5. Methods of series summation (4*)	95
Hints	98
Suggestions, solutions, and answers	98
6. Examples of transcendental numbers	99
6.A. Introduction (1)	99
6.B. Problems (3*)	100
6.C. Proof of Liouville's Theorem (2)	101
6.D. Simple proof of Mahler's Theorem (3*)	102
Chapter 7. Functions	105
1. The graph and number of roots of a cubic polynomial	105
1.A. Introduction	105
1.B. Problems	106
Hints	107
1.C. Statements of the main results	107
1.D. Proofs	109

2. Introductory analysis of polynomials (2)	112
Hints	114
3. The number of roots of a polynomial (3*)	115
Hints	117
Suggestions, solutions, and answers	117
4. Estimations and inequalities (4*) <i>By V. A. Senderov</i>	118
Suggestions, solutions, and answers	119
5. Applications of compactness (4*) <i>By A. Ya. Kanel-Belov</i>	119
Suggestions, solutions, and answers	121
 Chapter 8. Solving algebraic equations	 123
1. Introduction and statement of results	123
1.A. What is this chapter about?	123
1.B. Constructibility (1)	125
1.C. Insolvability in real radicals	126
1.D. Insolvability in complex radicals (2)	128
1.E. What is special about our proofs	130
1.F. Historical comments	131
1.G. Constructions with compass and straightedge (1)	132
Hints	133
2. Solving equations: Lagrange's resolvent method	133
2.A. Definition of expressibility in radicals of a polynomial (1)	133
2.B. Solution of equations of low degrees (2)	135
Suggestions, solutions, and answers	137
2.C. A reformulation of the constructibility in Gauss's Theorem (2)	139
Suggestions, solutions, and answers	140
2.D. Idea of the proof of constructibility in Gauss's Theorem (2)	140
2.E. Proof of the constructibility in Gauss's Theorem (3)	142
2.F. Efficient proofs of constructibility (4*)	143
Suggestions, solutions, and answers	148
3. Problems on insolvability in radicals	149
3.A. Representability using only one square root (1–2)	150
First hints	151
Suggestions, solutions, and answers	152
3.B. Multiple square root extractions (3*)	154
Suggestions, solutions, and answers	156
3.C. Representing a number using only one cube root (2)	158
Suggestions, solutions, and answers	159
3.D. Representing a number using only one root of prime order (3*)	162
Suggestions, solutions, and answers	163
3.E. There is only one way to solve a quadratic equation (2)	165
Suggestions, solutions, and answers	167
3.F. Insolvability "in real polynomials" (2)	168

Suggestions, solutions, and answers	170
3.G. Insolvability “in polynomials” (3)	170
Suggestions, solutions, and answers	171
3.H. Insolvability in complex numbers (4*)	172
3.I. Expressibility with a given number of radicals (4*)	173
4. Proofs of insolvability in radicals	175
4.A. Fields and their extensions (2)	175
4.B. Insolvability “in real polynomials” (3)	176
4.C. Insolvability “in polynomials” (3)	177
4.D. Non-constructibility in Gauss’s Theorem (3*)	179
4.E. Insolvability “in real numbers”	181
4.F. Insolvability “in numbers” (4*)	182
4.G. Kronecker’s Theorem (4*)	184
4.H. The real analogue of Kronecker’s Theorem (4*)	187
Bibliography	189
Index	195