Foreword

Problems, exercises, circles, and olympiads

This is a translation of Chapters 17–25 of the book *Mathematics Through Problems* by Mikhail B. Skopenkov and Alexey A. Zaslavsky and is part of the AMS/MSRI Mathematical Circles Library series. The goal of this series is to build a body of works in English that helps to spread the "Math Circle" culture.

A *mathematical circle* is an Eastern European notion. Math circles are similar to what most Americans would call a math club for kids, but with several important distinguishing features.

First, they are *vertically integrated*: young students may interact with older students, college students, graduate students, industrial mathematicians, professors, even world-class researchers, all in the same room. The circle is not so much a classroom as a gathering of young initiates with elder tribespeople, who pass down *folklore*.

Second, the "curriculum," such as it is, is dominated by *problems*, rather than specific mathematical topics. A problem, in contrast to an exercise, is a mathematical question that one doesn't know how, at least initially, to approach. For example, "What is 3 times 5?" is an exercise for most people, but it is a problem for a very young child. Computing 5^{34} is also an exercise, conceptually very much like the first, certainly harder, but only in a "technical" sense. And a question like "Evaluate $\int_2^7 e^{5x} \sin 3x dx$" is also an exercise—for calculus students—a matter of "merely" knowing the right algorithm and how to apply it.

Problems, by contrast, do not come with algorithms attached. By their very nature, they require *investigation*, which is both an art and a science, demanding technical skill along with focus, tenacity, and inventiveness. Math circles teach students these skills, not with formal instruction, but by *doing math* and observing others doing math. Students learn that a problem worth solving may require not minutes, but possibly hours, days, or even years of effort. They work on some of the classic folklore problems and discover how these problems can help them investigate other problems. They learn how not to give up and how to turn errors or failures into opportunities for more investigation. A child in a math circle learns to do exactly what a research mathematician does; indeed, he or she does independent research, albeit on a lower level, and often—although not always—on problems that others have already solved.

Finally, many math circles have a culture similar to a sports team, with intense camaraderie, respect for the "coach," and healthy competitiveness (managed wisely, ideally, by the leader/facilitator). The math circle culture is often complemented by a variety of problem-solving contests, often called *olympiads*. A mathematical olympiad problem is, first of all, a genuine problem (at least for the contestant) and usually requires an answer which is, ideally, a well-written argument (a "proof").

Why this book and how to use it

The Math Circles Library editorial board chose to translate Skopenkov and Zaslavsky's work from Russian into English because this book has an audacious goal—promised by its title—to develop mathematics through problems. This is not an original idea, nor just a Russian one. American universities have experimented for years with IBL (inquiry-based learning) and Moore-method courses, structured methods for teaching advanced mathematics through open-ended problem solving.[1]

But the authors' massive work is an attempt to curate sequences of problems for secondary students (their stated focus is high school students, but that can be broadly interpreted) that allow them to discover and recreate much of "elementary" mathematics (number theory, polynomials, inequalities, calculus, geometry, combinatorics, game theory, probability) and start edging into the sophisticated world of group theory, Galois theory, etc.

The book is impossible to read from cover to cover, nor should it be. Instead, the reader is invited to start working on problems that he or she finds appealing and challenging. Many of the problems have hints and solution sketches, but not all. No reader will solve all the problems. That's not the point—it is not a contest. Furthermore, some of the problems are not supposed to be solved but should be pondered. For example, Section 6 of Chapter 6 explores the unexpected connection between electrical circuits and random walks. In Chapter 7, the reader is encouraged to use similar ideas to analyze completely unrelated problems—dissecting squares into similar rectangles. Just because it is "too advanced" doesn't mean that it shouldn't be thought about!

Indeed, this is the philosophy of the book: mathematics is not a sequential discipline, where one is presented with a definition that leads to a lemma which leads to a theorem which leads to a proof. Instead it is an adventure, filled with exciting side trips as well as wild goose chases. The adventure is

[1] See, for example, `https://en.wikipedia.org/wiki/Moore_method` and `http://www.jiblm.org`.

its own reward, but it also, fortuitously, leads to a deep understanding and appreciation of mathematical ideas that cannot be accomplished by passive reading.

English-language references

Most of the references cited in this book are in Russian. However, there are many excellent books in English (some translated from Russian). Here is a very brief list, organized by topic. There are two bibliographies in this book. The references cited below are in the main bibliography at the end of the book.

Problem collections: *The USSR Olympiad Problem Book* [**SC**] is a classic collection of carefully discussed problems. Additionally, [**FKh**] and [**FBKYa1**, **FBKYa2**] are good collections of olympiads from Leningrad and Moscow, respectively. See also the collection of fairly elementary Hungarian contest problems [**Kur1**,**Kur2**,**Liu**] and the more advanced (undergraduate-level) Putnam Exam problems [**KKPV**].

Inequalities: See [**St**] for a comprehensive guide and [**AS**] for a more elementary text.

Geometry: *Geometry Revisited* [**CoxGr**] is a classic, and [**Chen**] is a more recent and very comprehensive guide to "olympiad geometry."

Polynomials and theory of equations: See [**B**] for an elementary guide and [**Bew**] for a historically motivated exposition of constructability and solvability and unsolvability.

Combinatorics: The best book in English, and possibly any language, is *Concrete Mathematics* [**GKP**].

Functions, limits, complex numbers, calculus: The classic book *Problems and Theorems in Analysis* by Pólya and Szegő [**PS**] is—like the current text—a curated selection of problems, but at a much higher mathematical level.

Paul Zeitz
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Chapter 1

Counting

This introductory chapter focuses on the question, “How many objects are there with given properties?” For further study, we recommend Chapter 1 of [**GDI**].

1. How many ways? (1)
By A. A. Gavrilyuk and D. A. Permyakov

1.1.1. (a) Call a positive integer *nice* if it contains only even digits. Write down all the nice two-digit numbers; how many are there?

(b) How many five-digit numbers are nice?

(c) How many six-digit numbers have at least one even digit?

(d) Which are there more of: seven-digit numbers that contain a 1 or seven-digit numbers that have no 1’s?

1.1.2. We wish to form a committee of eight people, chosen from two mathematicians and ten economists. In how many ways can this be done if at least one mathematician is to be included on the committee?

1.1.3. (a) Find the sum of all seven-digit numbers that can be obtained by permuting the digits $1, \ldots ,7$.

(b) Find the sum of all four-digit numbers with no zeros and no repeating digits.

(c) Find the sum of all four-digit numbers that do not contain repeating digits.

1.1.4. (a) Black and white kings occupy two squares of a chessboard. The player moves a king (alternating between black and white king) to an adjacent square (horizontally, vertically, or diagonally). The kings are friends, so they can occupy neighboring squares but not be in the the same square. Is it possible to get the kings to occupy every pair of squares (s_1, s_2) with the white king on s_1 and the black king on s_2, exactly once?

(b) The same question, but now the kings are unable to move diagonally.

1.1.5. (a) Find the sum of all six-digit numbers obtained by all permutations of the digits $4, 5, 5, 6, 6, 6$.

(b) Find the sum of all ten-digit numbers obtained by all permutations of the digits $4, 5, 5, 6, 6, 6, 7, 7, 7, 7$.

1.1.6. (a) Tom Sawyer was commissioned to paint 8 boards of a fence white. He is lazy, so he will paint no more than 3 boards. In how many ways can he do this?

(b) How many ways are there to paint no more than 5 boards?

(c) How many ways are there to paint any number of boards?

Suggestions, solutions, and answers

1.1.1. *Answers*: (b) 2500; (c) 884 375; (d) the numbers containing a 1.

(b) *Solution* (A. Kolochenkov). The first digit of a nice number can be 2, 4, 6, or 8, so there are just 4 options. For each digit in the second to fifth place, there are 5 options: 0, 2, 4, 6, 8. So the total amount of nice numbers is $4 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 2500$.

This reasoning is called the *product rule* in combinatorics and is discussed in detail in [**Vil71b**].

(c) *Solution* (A. Kolochenkov). Subtract from the total number of six-digit numbers the number of six-digit numbers consisting entirely of odd digits. Then there will remain numbers in which at least one digit is even. Since there are only five odd digits, the product rule yields $9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 - 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 884\,375$.

This reasoning (the total quantity of numbers is equal to the sum of the quantity of numbers from odd digits and the quantity of numbers with even digit) is called the *sum rule* and is discussed in detail in article [**Vil71b**].

1.1.2. *Answer*: 450.

1.1.3. *Answers*: (a) 22 399 997 760; (b) 16 798 320; (c) 24 917 760.

(a) *Solution* (T. Cherganov). There are 7! permutations of seven digits. Each digit will occur in all positions the same number times, equal to $\frac{7!}{7} = 6!$. Then the sum of the digits in each position is equal to $6! \cdot (1 + 2 + \cdots + 7) = 20160$. So the sum of all numbers is equal to

$$20160 + 20160 \cdot 10 + 20160 \cdot 100 + \cdots + 20160 \cdot 1000000 \\ = 20160 \cdot 1111111 = 22399997760.$$

(b) *Solution* (S. Kudrya). We calculate the sum of the digits in each position. Each digit is included in the sum $8 \cdot 7 \cdot 6$ times; indeed, for the first of the remaining positions there are 8 possibilities to place a digit, for the second one there are 7, for the third there are 6. Therefore, the amount in each position is $8 \cdot 7 \cdot 6 \cdot 1 + 8 \cdot 7 \cdot 6 \cdot 2 + 8 \cdot 7 \cdot 6 \cdot 3 + \cdots + 8 \cdot 7 \cdot 6 \cdot 9 = 8 \cdot 7 \cdot 6 \cdot (1 + 2 + 3 + \cdots + 9) = 8 \cdot 7 \cdot 6 \cdot \frac{9 \cdot 10}{2}$ =15120. Multiplying this by 1111 produces the desired sum of numbers.

1.1.4. (a) *Answer*: yes. *Hint*: explicitly construct an example.

(b) *Answer*: no. *Hint*: count the number of positions.

Suggestion. Suppose they can. The number of “one-color” positions for which the kings stand on the cells of the same color is $64\cdot 31$. The number of “multi-colored” positions is $64\cdot 32$. The type of position flips after each move of a king (they cannot move diagonally). If all these positions appear once, then quantities $64\cdot 32$ and $64\cdot 31$ differ by no more than one. Contradiction.

1.1.5. *Answers*: (a) 35 555 520; (b) 83 999 999 991 600.

(a) Six digits can be arranged in a row in $6! = 720$ ways. However, since the sixes are indistinguishable and the fives are indistinguishable, the total number of distinguishable rearrangements of these six digits is $\frac{6!}{3!\cdot 2!} = 60$. Each digit occurs an equal number of times in each of the six positions. The four occurs 60 times, which means that in each position it occurs 10 times. Five occurs 120 times, which means that in each position it occurs 20 times. The six occurs 180 times, which means that in each position it occurs 30 times. The sum of the digits in each position is $4\cdot 10 + 5\cdot 20 + 6\cdot 30 = 320$. So the sum of all numbers is $320 + 320\cdot 10 + 320\cdot 100 + 320\cdot 1000 + 320\cdot 10\,000 + 320\cdot 100\,000 = 320\cdot 111\,111$.

(b) *First solution* (V. Tsepelev). We will count the number of occurrences of each of the digits in a specific position. Having fixed one of the digits, we may freely arrange the digits in the remaining nine. Note that 5 occurs twice, 6 three times, and 7 four times. Permutations of identical digits give identical numbers, so 4 will occur $\frac{9!}{2!3!4!}$ times. Similarly, 5 will occur $\frac{9!}{1!3!4!}$ times, and 6 and 7 will occur $\frac{9!}{1!2!2!4!}$ and $\frac{9!}{1!2!3!3!}$ times.

Now we compute the sum of the digits in each position: $4\cdot\frac{9!}{2!3!4!} + 5\cdot\frac{9!}{3!4!} + 6\cdot\frac{9!}{2!2!4!} + 7\cdot\frac{9!}{2!3!3!} = 75\,600$. Finally, multiply by 111 111 111.

Second solution (V. Tsepelev). The answer can be obtained faster if you notice that the average of all digits of the set $4, 5, 5, 6, 6, 6, 7, 7, 7, 7$ is equal to 6. Therefore, the sum will be the same as for the same quantity of repeats of 6 666 666 666. The total number of repeats is $\frac{10!}{1!2!3!4!}$, so the sum is $6\,666\,666\,666\cdot\frac{10!}{1!2!3!4!} = 83\,999\,999\,991\,600$.

1.1.6. (a) *Answer*: 93.

Suggestion. If Tom paints no more than three boards, then he can paint $0, 1, 2$, or 3 boards. He can do it in $\binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \binom{8}{3} = 93$ ways.

(b) *Answer*: 219.

Similarly, no more than 5 boards can be painted in $\binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \binom{8}{3} + \binom{8}{4} + \binom{8}{5} = 219$ ways.

(c) *Answer*: 256 $\left(= 2^8 = \binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \binom{8}{3} + \binom{8}{4} + \binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8}\right)$.

2. Sets of subsets (2) *By D. A. Permyakov*

1.2.1. Every evening, Uncle Chernomor selects 9 or 10 bogatyrs (hero warriors) from his company of 33 for guard duty. What are the fewest number of evenings such that it is possible that each of the bogatyrs went on duty for the same number of times?

1.2.2. A classroom has 33 students. Each student was asked how many students there are with the same first name as his in the class and how many students there are with the same last name as his (including relatives). It turned out that all integers from 0 to 10, inclusive, were reported. Must the classroom contain two students with the same first and last names?

1.2.3. (a) What is the maximum number of pairwise intersecting subsets that can be chosen from a set of 100 elements?

(b) In how many ways can a set of n elements be decomposed into 2 subsets?

(c) How many different unordered pairs of disjoint subsets can be chosen from a set of n elements?

1.2.4. One is given 2007 sets, each of which contains 40 elements. Any two of these sets have exactly one common element. Is there necessarily an element belonging to each of these sets?

1.2.5. In the set consisting of 100 elements, choose 101 distinct three-element subsets. Must there exist two subsets among them that have exactly one common element?

1.2.6. A questionnaire was conducted in a country in which the respondent was required to name his favorite writer, artist, and composer. It turned out that each writer, artist, or composer was mentioned as a favorite by no more than k people. Is it true that all of the surveyed people can always be divided into no more than $3k-2$ groups, so that in each group any two people have completely different tastes?

1.2.7. Consider all nonempty subsets of the set $\{1, 2, \dots, N\}$ that do not contain consecutive numbers. For each subset, calculate the product of its elements. Find the sum of the squares of these products.

Suggestions, solutions, and answers

1.2.1. *Answer*: 7.

Suggestion. Let 9 warriors be on duty for $m \geq 0$ days, and let 10 warriors be on duty for $n \geq 0$ days. Then (since each of the warriors went on duty the same number of times) the number $9m + 10n$ must be divisible by 33. Case by case analysis of all possible values of n shows that $9m + 10n = 33$ has no solutions in nonnegative integers. However, $9m + 10n = 66$ has the solution $m = 4$, $n = 3$. It is easy to construct an example spanning $m + n = 7$

days where each warrior serves exactly 2 times. If $9m + 10n \geq 99$, then $m + n \geq \frac{99}{10} > 7$. Therefore, the minimum number of days is 7.

1.2.2. *Answer*: yes.

Solution (T. Cherganov). Divide the students into groups by first and last name. Then each student will fall into exactly two groups. There are at least 11 such groups, since all numbers from 0 to 10 inclusive were reported. The total number of people in the 11 groups is not less than $\frac{1+2+\dots+11}{2} = 33$. This means that there are exactly 11 groups, and the number of people in them is $1, 2, \dots, 11$, respectively. Consider the group of 11 students. Without loss of generality, assume that 11 people have the same last name. Each of them is a member of one of the remaining 10 groups (by first name). By the pigeonhole principle, at least one of these groups (by first name) includes 2 people that have the same last name.

1.2.3. (a) *Answer*: 2^{99}.

Suggestion. The set has 2^{100} subsets. Divide them in pairs: put each subset in a pair with its complement. Then there will be a total of 2^{99} pairs, and from each pair we can choose no more than one subset. To find 2^{99} pairwise intersecting subsets, just take all subsets containing some fixed element.

(b)*Answer*: 2^{n-1}.

Consider the partition of the set into two subsets to be the choice of some subset and its complement. Choosing a subset, we can either select or not select a particular element. So the number of ways to choose a subset is 2^n. But we counted each partition twice, since we can reverse the role of "subset" and "complement". Therefore, there are 2^{n-1} different partitions.

(c)*Answer*: $\frac{3^n-1}{2}$.

Suggestion. First calculate the number of ways to select an ordered pair of disjoint subsets. There are three options for each element: it either lies in the first subset, the second one, or neither subset. There will be 3^n such ordered pairs. An unordered pair of two empty subsets corresponds to one ordered pair of empty subsets. Any other unordered pair corresponds to two ordered pairs of subsets. This means that a total number of unordered pairs is $\frac{3^n-1}{2}$.

1.2.4. *Answer*: yes.

Solution (V. Tsepelev). We denote given sets $J_1, \dots, J_{2007}$. Let $J_1 = (a_1, \dots, a_{40})$. Split the numbers corresponding to the indices in the family $J_2, \dots, J_{2007}$ into sets of the form $X_i = \{j \geq 2 \colon a_i \in J_j\}$, $i = 1, \dots, 40$, since each pair of sets has exactly one common element.

Since $\frac{2007-1}{40} = 50.15$, one of X_i—without loss of generality, suppose it is X_1—contains at least 51 elements. Consider two cases. If $X_1 = \{2, \dots, 2007\}$, then the statement is proved (a_1 belongs to all sets; there are no other common elements). If not, then assume without loss of generality that $2007 \notin X_1$ and that $2, \dots, 52 \in X_1$. Then for all $j = 2, \dots, 52$ the intersection $X_j \cap X_{2007}$ will contain exactly one element, and this element is diferent for different indices j (since any two sets have exactly one common element). But this is not possible because X_{2007} consists of only 40 elements, which is less than 51.

1.2.5. *Answer*: yes.

1.2.6. *Answer*: yes.

1.2.7. *Answer*: $(N+1)! - 1$.

3. The principle of inclusion-exclusion (2)
By D. A. Permyakov

This section is devoted to the proof and application of the inclusion-exclusion formula, also known as the principle of inclusion-exclusion. It allows you to answer the question, "How many objects are there with given properties?" in many difficult cases. It will require basic skills for solving combinatorics problems. In particular, one must be able to give rigorous proofs using one-to-one correspondences and the rules of sums and products. For example, it is useful to solve problems from Section 1 of this chapter or problems from article [**Vil71b**].

1.3.1. In how many ways can you rearrange the numbers from 1 to n so that

(a) neither 1 nor 2 occurred in its original position;
(b) exactly one of the numbers 1, 2, and 3 stayed in its original position;
(c) none of the numbers 1, 2, and 3 occurred in their original positions;
(d) none of the numbers 1, 2, 3, and 4 occurred in its original position?

Define the Euler function $\varphi(n)$ to be the number of integers between 1 and n relatively prime to the number n.

1.3.2. (a) Find the number of integers from 1 to 1001 not divisible by any of the numbers 7, 11, 13.

(b) Find $\varphi(1)$, $\varphi(p)$, $\varphi(p^2)$, $\varphi(p^\alpha)$, where p is a prime, $\alpha > 2$.

(c) Prove that $\varphi(n) = n\left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_s}\right)$, where $n = p_1^{\alpha_1} \cdot \dots \cdot p_s^{\alpha_s}$ is the canonical decomposition of the number n into distinct primes p_k.

1.3.3. (a) On the floor of a room with area $24\,\mathrm{m}^2$ there are three area rugs (of arbitrary shape) each with area $12\,\mathrm{m}^2$. Then there exist two rugs with the area of the intersection at least $4\,\mathrm{m}^2$.

(b) There are five patches (of arbitrary shape) on a caftan.[1] The area of each patch is more than three-fifths of the area of caftan. Then there are two patches such that the area of their intersection is more than one-fifth the area of the caftan.

(c)* Same as in part (b), but the area of each patch is assumed to be greater than *half* the area of the caftan.

[1] A *caftan* is an old-fashioned Russian cloth similar to a jacket.

In this section, we encounter problems of the following type: a finite set U and a set of properties (subsets) $A_k \subset U$, $k = 1, \dots, n$, are given. We wish to find the number of elements for which at least one of the A_k properties is satisfied (i.e., $|A_1 \cup \dots \cup A_n|$), or the number of elements for which none of the properties A_k is satisfied (i.e., $|U - (A_1 \cup \dots \cup A_n)|$). For this, two versions of the inclusion-exclusion principle are used (see problem 1.3.5(b)). Moreover, if in all intersections of the sets of the family the number of elements depends only on the number of intersected sets, then the formula can be simplified (see problem 1.3.5(a)).

1.3.4. Consider the subsets A_1, A_2, A_3, A_4 of a finite set U. Prove the following equalities:

(a) $A_1 \cup A_2 = (A_1 \setminus A_2) \sqcup (A_1 \cap A_2) \sqcup (A_2 \setminus A_1)$;

(b) $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$;

(c) $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$.

(d) The number of elements in U that do not belong to any of the subsets A_1, A_2, A_3 is

$$|U| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3| - |A_1 \cap A_2 \cap A_3|.$$

(e) For $k = 1, 2, 3, 4$ define

$$M_k := \sum_{1 \le i_1 < \dots < i_k \le 4} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}|.$$

Prove that the number of elements in A that do not belong to any of A_i is equal to $|U| - M_1 + M_2 - M_3 + M_4$.

(f) Using the above notation, the number of elements belonging to exactly one of the sets A_i is $M_1 - 2M_2 + 3M_3 - 4M_4$.

1.3.5. Inclusion-exclusion principle. Consider the subsets $A_1, \dots, A_n$ of a finite set U. By definition set $\bigcap\limits_{j \in \emptyset} A_j := U$.

(a) Suppose that the number $\alpha_{|S|} := \left| \bigcap\limits_{j \in S} A_j \right|$ depends only on the size $|S|$ of the set $S \subset \{1, \dots, n\}$ of indices, but not on the set itself. Then

$$|A_1 \cup \dots \cup A_n| = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \alpha_k,$$

$$|U - (A_1 \cup \dots \cup A_n)| = \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \alpha_k.$$

(b) Define $M_k := \sum_{S\in\binom{n}{k}} \left|\bigcap_{j\in S} A_j\right|$, where the indicated summation is over all k-element subsets of the set $\{1,\dots,n\}$. In particular, $M_0 := |U|$. Then

$$|A_1 \cup \dots \cup A_n| = M_1 - M_2 + M_3 - \dots + (-1)^{n+1} M_n,$$
$$|U - (A_1 \cup \dots \cup A_n)| = M_0 - M_1 + M_2 + \dots + (-1)^n M_n.$$

(c) **Bonferroni Inequalities.** For any $0 \le s < n/2$, the following inequalities hold:

$$\begin{aligned} M_1 - M_2 + M_3 - \dots - M_{2s} &\le |A_1 \cup \dots \cup A_n| \\ &\le M_1 - M_2 + M_3 - \dots + M_{2s+1}, \\ M_0 - M_1 + M_2 - \dots + M_{2s} &\ge |U - (A_1 \cup \dots \cup A_n)| \\ &\ge M_0 - M_1 + M_2 - \dots - M_{2s+1}. \end{aligned}$$

(d) The number of elements belonging to exactly r from the subsets $A_1, \dots, A_n$ is equal to $\sum_{k=r}^{n} (-1)^{k-r} \binom{k}{r} M_k$.

1.3.6. A shelf holds 10 different books.

(a) In how many ways can they be rearranged so that none of the books stays in place?

(b) Prove that the number of rearrangements for which exactly 4 books stay in place is more than 50 000.

In the next problem, the answer can be in terms of sums (similar to the inclusion-exclusion principle).

1.3.7. (a) In how many ways can you arrange 20 tourists into 5 different houses so that not a single house is empty?

(b) How many different surjections $f\colon \mathbb{Z}_k \to \mathbb{Z}_n$ are there?

1.3.8. The numbers $1, 2, \dots, n$ are placed on a circle. Find the number of ways to select k of them so that no two selected numbers were adjacent.

(b) Find the number of ways to seat n pairs of warring knights at a round table with numbered seats so that no two warring knights sit next to one another.

1.3.9. A cube with edge of length 20 is divided into 8000 unit cubes, and each unit cube is given a numerical label. It is known that in each row of 20 cubes parallel to an edge of the cube, the sum of the labels is 1 (rows in all three directions are considered). The label 10 is used on at least one cube. Three layers of $1 \times 20 \times 20$ pass through this cube parallel to the faces of the cube. Find the sum of all the labels not contained in any of these layers.

1.3.10.* How many six-digit numbers are there with no two 7's adjacent and with

(a) no more than three 7's;
(b) not more than four 7's;
(c) any number of 7's?

1.3.11.* Prove the following formula:

$$n! \cdot x_1 x_2 \dots x_n = (x_1 + x_2 + \dots + x_n)^n - \sum_{1 \le i_1 < i_2 < \dots < i_{n-1} \le n} (x_{i_1} + x_{i_2} + \dots + x_{i_{n-1}})^n + \sum_{1 \le i_1 < i_2 < \dots < i_{n-2} \le n} (x_{i_1} + x_{i_2} + \dots + x_{i_{n-2}})^n - \dots + (-1)^{n-1} \sum_{i=1}^{n} x_i^n.$$

Suggestions, solutions, and answers

1.3.1. (a)*Answer*: $n! - 2(n-1)! + (n-2)!$.

Suggestion. There are $n!$ ways in total to rearrange our numbers. Subtract from them the $(n-1)!$ permutations for which the number 1 remains in place. Subtract also the $(n-1)!$ permutations for which the number 2 remains in place. However, there are $(n-2)!$ permutations for which both numbers 1 and 2 remain in place, which we "subtracted twice". Thus, the number $(n-2)!$ needs to be added, yielding $n! - 2(n-1)! + (n-2)!$.

Comment. This solution is formalized by the principle of inclusion-exclusion, problem 1.3.5, which we assume you have proved and will use in subsequent problems.

1.3.2. (a)*Answer*: 720.

First suggestion. For each divisor j of 1001, let A_j denote the set of numbers from 1 to 1001 divisible by j. Then

$$|A_j| = \frac{1001}{j} \quad \text{and} \quad A_{p_1} \cap \dots \cap A_{p_k} = A_{p_1 \cdot \ldots \cdot p_k}$$

for distinct primes $p_1, \dots, p_k$. Therefore, we wish to compute

$$\begin{aligned}
&|\{1, \dots, 1001\} - (A_7 \cup A_{11} \cup A_{13})| \\
&\quad = 1001 - |A_7| - |A_{11}| - |A_{13}| + |A_7 \cap A_{11}| + |A_7 \cap A_{13}| \\
&\qquad + |A_{11} \cap A_{13}| - |A_7 \cap A_{11} \cap A_{13}| \\
&\quad = 1001 - |A_7| - |A_{11}| - |A_{13}| + |A_{77}| + |A_{91}| + |A_{143}| - |A_{1001}| \\
&\quad = 1001 - 143 - 91 - 77 + 7 + 11 + 13 - 1 = 720.
\end{aligned}$$

Second suggestion. Use the equality $\varphi(1001) = \varphi(7)\varphi(11)\varphi(13)$.

(c) *Suggestion.* For any $j \mid n$, define A_j as a subset of the set $\{1, \dots, n\}$, consisting of numbers that are divisible by j. Clearly

$$|A_j| = \frac{n}{j} \quad \text{and} \quad A_{p_1} \cap \dots \cap A_{p_k} = A_{p_1 \cdot \dots \cdot p_k}$$

for distinct primes $p_1, \dots, p_k$.

Define

$$M_k := \sum_{1 \le j_1 < \dots < j_k \le s} \frac{n}{p_{j_1} p_{j_2} \cdot \dots \cdot p_{j_k}} \quad \text{for } k \ge 1 \text{ and } M_0 := n.$$

By definition, $\varphi(n) = |\{1, \dots, n\} - (A_{p_1} \cup \dots \cup A_{p_s})|$, whence the inclusion-exclusion principle (1.3.5(b)) yields

$$\varphi(n) = M_0 - M_1 + M_2 - \dots + (-1)^s M_s = n\Big(1 - \frac{1}{p_1}\Big) \cdots \Big(1 - \frac{1}{p_s}\Big).$$

1.3.3. (a) Let A_j denote the set of points covered by the jth rug, and let $|A_j|$ be the area of the jth rug.

Suppose the statement is false; i.e., for any distinct j, k we have $|A_j \cap A_k| < 4$. Consequently,

$$\begin{aligned} 24 &\ge |A_1 \cup A_2 \cup A_3| \\ &\ge |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| \\ &> 12 + 12 + 12 - 4 - 4 - 4 = 24, \end{aligned}$$

where the second inequality follows from the analogue of the Bonferroni inequality for areas (1.3.5(c)), which produces a contradiction.

(b), (c) These problems are discussed in detail in the article [**Yag74**].

1.3.6. (a) *Answer*: $1\,334\,961 = \mathrm{round}(10!/e)$, where

$$e := 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + \cdots$$

and $\mathrm{round}(x)$ is the nearest integer to a real number x.

Suggestion. Set $n := 10$. Let U be the set of all book permutations, and let A_j be the set of book permutations for which the jth book remains in its original place. Consider an arbitrary k-element subset $S \subset \{1, \dots, n\}$. Then $\bigcap_{j \in S} A_j$ consists of those permutations of books for which each of the books $j \in S$ remains in its original place. Therefore

$$\Big|\bigcap_{j \in S} A_j\Big| = (n-k)(n-k-1)\dots 1 = (n-k)!.$$

Using the principle of inclusion-exclusion, we obtain

$$\begin{aligned} |U - (A_1 \cup \dots \cup A_n)| &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! \\ &= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \dots + (-1)^n \frac{n!}{n!} = \mathrm{round}(n!/e). \end{aligned}$$

(b) To construct the desired permutation, select those 4 books from 10 that remain in their original places, and then rearrange the remaining books, so that no book remains in its original place. Using (a), the required quantity is equal to

$$\binom{10}{4}\Big(6! - \frac{6!}{1!} + \frac{6!}{2!} - \frac{6!}{3!} + \dots + \frac{6!}{6!}\Big)$$
$$> \frac{10\cdot 9\cdot 8\cdot 7}{24}\Big(\frac{6!}{2} - \frac{6!}{6}\Big) = 210\cdot 240 = 7\cdot 7200 > 50\,000.$$

1.3.7. (a) Let U be the set of all tourist arrangements, and let A_j be the set of tourist arrangements for which the jth house is empty.

The number of arrangements for which all houses with numbers from the set S are empty is equal to $\Big|\bigcap\limits_{j\in S} A_j\Big| = (5-|S|)^{20}$. Inclusion-exclusion yields

$$|U-(A_1\cup\dots\cup A_5)| = \sum_{k=0}^{5}(-1)^k\binom{n}{k}(5-k)^{20}$$
$$= 5^{20} - \binom{5}{1}4^{20} + \binom{5}{2}3^{20} - \binom{5}{3}2^{20} + \binom{5}{4}1^{20}.$$

(b) Let $U = \mathbb{Z}_n^{\mathbb{Z}_k}$ be the set of all mappings from $\mathbb{Z}_k$ to $\mathbb{Z}_n$. For each $j = 1,\dots,n$, let $A_j = (\mathbb{Z}_n - \{j\})^{\mathbb{Z}_k}$ be the set of all mappings $\mathbb{Z}_k$ to $\mathbb{Z}_n$, whose image does not contain j. Then the set of surjections from $\mathbb{Z}_k$ to $\mathbb{Z}_n$ is the set $U-(A_1\cup A_2\cup\dots\cup A_n)$. Observe that $\Big|\bigcap\limits_{j\in S} A_j\Big| = (n-|S|)^k$. Inclusion-exclusion yields

$$|U-(A_1\cup A_2\cup\dots\cup A_n)|$$
$$= n^k - \binom{n}{1}(n-1)^k + \binom{n}{2}(n-2)^k - \dots + \binom{n}{n}\cdot 1^k.$$

1.3.8. (a) *Answer*: $\frac{n}{n-k}\binom{n-k}{k}$.

(b) Use part (a).

1.3.9. *Answer*: 333.

First suggestion. Inclusion-exclusion yields $400-3\cdot 20+3\cdot 1-10 = 333$.

Second suggestion (from site `http://problems.ru`). One horizontal layer G and two vertical layers pass through a given unit cube K. The sum of all numbers in the 361 vertical columns that are not in the other two vertical layers is 361. Then we must subtract the sum S of labels of the 361 cubes lying in the intersection of these columns with G. These cubes are completely covered by 19 columns lying in G. The sum of all the labels in these columns (it is equal to 19) exceeds S by the sum of the 19 labels lying in the column containing K that is perpendicular to them. The last amount is obviously $1-10=-9$. Hence $S = 19-(-9) = 28$. Our final sum is thus $361-28 = 333$.