# Preface

The purpose of these seminars is to explore the riches of geometry. The topics are designed to clarify and enhance the understanding you have of the subject. Essentially all of the familiar facts from high school geometry appear here. However, they are presented in ways that make connections to other geometric concepts and provide material for classroom use. We draw on the pictorial nature of geometry, for this is what attracts students at every level to the subject.

The seminars are designed to encourage dialogues between the seminar leader and the other participants. Your input is important. The sessions are conversations, not lectures, in geometry.

Here is a brief outline of the topics of the ten geometry seminars along with some of the special features of each. Classroom activities, which are not for the most part noted below, are dispersed throughout each seminar.

• Seminar 1, **The Geometry of Polygons in the Plane**, contains a very careful definition of "polygon" with numerous examples, as well as a review of the basic concepts associated with a polygon. Classification trees for triangles and for quadrilaterals are constructed. The concepts of congruence of figures and of angles are defined.

• Seminar 2, **More Fundamentals of Plane Geometry**, introduces the notion of triangulation, one of the fundamental techniques of geometry. Triangulation is used here to derive the formula for the sum of the degrees in the interior angles of a polygon. Regular polygons are defined and the formula for the number of degrees in each interior angle of a regular polygon is deduced. A proof of the Pythagorean theorem is given by introducing an important diagram of squares and triangles which also plays a role in Seminar 7. The notion of distance between two points in the plane as well as the distance formula are examined and then used to give the definition and the equation of a circle. The seminar ends with a review of the straightedge and compass construction of the perpendicular bisector of a line segment and an interesting classroom activity on construction of a circle through three noncollinear points.

• Seminar 3, **Tessellation**, is an example of "geometry in action." Given a shape, the question for study here is whether there is a pattern, filling the plane with no gaps or overlaps, composed of copies of the shape. The question is answered by means of a series of "hands-on" activities, the first of which results in the fact that all triangles tessellate the plane, that is, for any triangle, there is a pattern, filling the plane with no gaps or overlaps, made up of copies of this triangle. A quite surprising fact presented is that all quadrilaterals tessellate the plane. These activities reinforce many of the basic geometric names and properties considered in earlier seminars. The seminar concludes with a brief summary of the history of tessellation.

• Seminar 4, **Regular Polygons and Regular Polyhedra**, begins with straightedge and compass constructions of regular $n$-gons, for $n = 3, 4, 5$ and 6. These constructions use the fact that a regular $n$-gon inscribed in a circle has a decomposition into $n$ isosceles triangles each of which has base angles and apex angle that are easily calculated. The second part of the seminar moves into 3-space with the constructions of the five regular polyhedra, also known as the Platonic solids. The questions of why there are exactly five regular polyhedra and why their faces are precisely the regular $n$-gons, for $n = 3$, 4 and 5, are studied.

• Seminar 5, **Symmetry**, considers certain motions of a polygon, called symmetries of the polygon. Symmetries are motions that preserve distance and angles and return the polygon to its original place in the plane. These are, in fact, its rotations and its reflections. The symmetries of an equilateral triangle are studied first, by means of models with numbered vertices. Composition of symmetries is explored, a multiplication table is constructed and the notion of a group is introduced. The representation of symmetries by permutations is presented, and composition, or multiplication, of permutations is defined. The seminar concludes with consideration of the symmetries of a square. With one important exception, which is discussed, the ideas involved for the square are natural generalizations of those for the symmetries of an equilateral triangle.

• Seminar 6, **Lattice Polygons**, highlights the special polygons that can be drawn in a lattice. The mathematical concept of a lattice is defined and it is noted that a geoboard is a practical realization of a lattice for the classroom. Lattice polygons are introduced and many examples are given. The question of whether an equilateral triangle can be constructed in a lattice is explored. Properties of lattice polygons, such as the important fact that the area of a lattice square is an integer, are discussed. The positive integers that are the areas of lattice squares are studied in detail. A significant fact presented is that a positive integer is the area of a lattice square if and only if it is the sum of two squares of nonnegative integers, not both equal to 0. This leads to an interesting classroom investigation of exactly which positive integers are a sum of two squares of nonnegative integers.

• Seminar 7, **The Area of Polygonal Regions**, discusses the four essential properties of area. The formulas for the area of a triangle and for the area of each of the "named" quadrilaterals are deduced. The relationship between these properties and the methods for introducing the concept of area in the classroom is examined. The use of triangulation to compute areas of other polygons is discussed and Heron's formula is introduced. For lattice polygons, Pick's theorem for the number of lattice points inside and on a lattice polygon is obtained by experimentation. The use of Pick's formula for computing the area of a lattice polygon is noted and the question, from Seminar 6, of whether an equilateral triangle can be constructed in a lattice is resolved.

• Seminar 8, **The Area of a Disk and Disk Packing**, studies the geometry of circles and its application to disk packing. To pack with disks means to cover or pack a specified area with as many disks as possible without overlaps. Two packings, namely, the square packing and the triangle packing, are defined, as is the density of a packing, which measures how good the packing or covering is. In classroom activities, several small spaces are packed with the square and the triangle packings of pennies, and the density of each is calculated. The density of these packings on the whole plane is calculated by applying the topics on circles discussed in the seminar. These include the definitions of the circumference of a circle and of the constant $\pi$, and the area of a disk. The circumference of a circle and the area of a disk are viewed as limits of the perimeters of inscribed regular polygons and of the areas of regular polygons, respectively.

• Seminar 9, **Dissection** is another example of "geometry in action." Dissection is the process of cutting up a polygon and then rearranging all of the pieces to form another polygon. Dissection was introduced in the discussion of area in Seminar 7, where it was seen that dissection of one polygon into another implies that the polygons have the same area. This fact motivates the investigation taken up in this seminar. It is the important question of whether, given two polygons of the same area, one can be cut up to form the other. The analysis of this problem proceeds by a series of dissections that include the dissection of a triangle into a parallelogram, the dissection of a parallelogram into a rectangle of the same base length and height, and the dissection of any rectangle into any other of the same area. For classroom activities, the participants take up scissors and try to carry out these dissections for particular polygons.

• Seminar 10, **Geometry in Three Dimensions**, begins with a survey of points, lines and planes, the basic geometric objects in 3-space. The concepts of parallelism and perpendicularity are studied. The fundamental technique of reducing three dimensional problems to familiar facts in plane geometry is explained and illustrated. Coordinates in 3-space are introduced. Two types of solids, namely, prisms and pyramids, are studied. The cube and regular tetrahedron constructed in Seminar 4 are examples of these. Particular attention is paid to two facts about their cross sections, namely that all cross sections of a prism are congruent to the base and that all cross sections of a pyramid are similar to the base. The final topic for the session is the volume of a prism and of a pyramid. Four properties for volume that correspond to the four properties for area are discussed. Formulas for the volume of a prism and for the volume of a pyramid are deduced, with the derivation of the latter pleasingly explicating the 1/3 that appears there.

These seminars are designed for use by teachers of geometry, undergraduate and graduate students preparing for the teaching profession, and educators conducting enrichment programs such as Math Circles and Young Scholars. Students of geometry may also find these notes interesting. It is assumed that the seminar participants have already taken a high school geometry course. Several high school geometry texts for reference are listed below, as well as some books for further exploration of the ideas contained herein.

## References for Basic Geometry

*Geometry*, first or second edition, by Harold R. Jacobs; Freeman, 1987.

*Geometry* by Edwin E. Moise and Floyd L. Downs, Jr.; Addison-Wesley, 1991.

*Mathematics for High School Teachers – An Advanced Perspective* by Zalman Usiskin, Anthony Peressini, Elena Anne Marchisotto and Dick Stanley: Prentice Hall, 2002.

## References for Further Exploration

*The Thirteen Books of Euclid's Elements: Translation with Introduction and Notes by T. L. Heath*, Vol. 1–3, second edition, by Euclid and T. L. Heath; Dover Publications, 1956.

*Foundations of Geometry* by David Hilbert; Kessinger Publishing, 2010.

*Elementary Geometry from an Advanced Standpoint* by Edwin E. Moise; Addison Wesley, 1990.

## Acknowledgements

We acknowledge with gratitude the assistance of John Boller, Jon Gleason and Alex Zorn. We thank the National Science Foundation for partial support, under Grant #NSF ESI-0101913, during the preparation of this manuscript.