Chapter 1

Seminar 1: Polygons in the Plane

Preface. To the Seminar Leaders

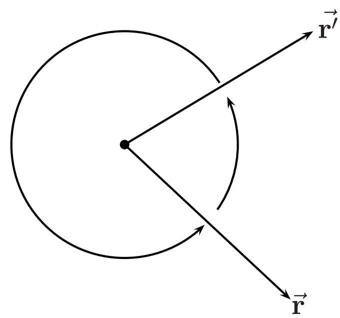
The first meeting with the seminar participants sets the tone for the remaining ones. It must present the subject and establish the conversational spirit of the seminars.

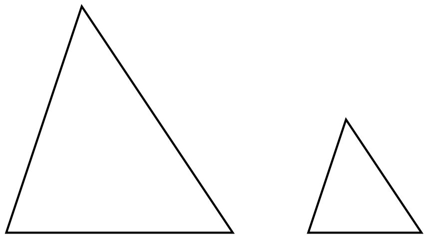
One approach that has been very successful over the years is to begin by examining words associated with geometry. Each participant is asked, in turn, for a geometric word or idea. No repeats are allowed, and simple, basic terms are encouraged. The words are recorded on the board. Here are some words, in no particular order, that have been suggested in past seminars.

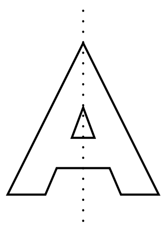
angle	hypotenuse	perpendicular	parallel
triangle	polygon	point	circle
distance	congruent	perimeter	area
similar	plane	absolute value	quadrilateral
volume	line	cube	symmetry
slope	transversal	coordinate	radius.

During the conversation that ensues as the list of words is prepared, participants are encouraged to find connections between the words. For homework, the participants are asked to draw a picture, unaccompanied by words, to illustrate each of the terms in the list. The purpose of the exercise is to place emphasis on the visual nature of geometry.

Here are some possible illustrations of the words angle, similar and symmetry.







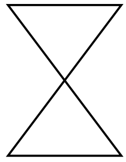
Now, we continue Seminar 1 with a discussion of polygons.

Polygons in the Plane

1. The Definition of a Polygon

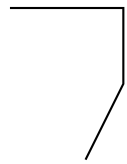
Before giving a formal definition of a polygon, we explore the geometric idea behind the definition. Almost everyone knows a polygon when they see one. Triangles, squares, rectangles, parallelograms, etc. are familiar examples of polygons. The sides of these polygons are *line segments*. So we see that a polygon is a collection of line segments.

Let's look at some collections of line segments.

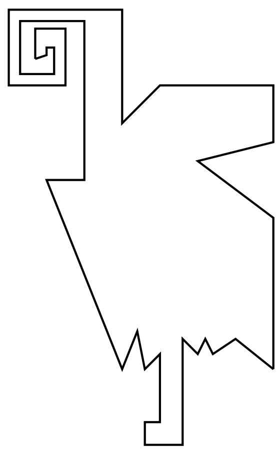


We do not want to consider this shape to be a polygon because, depending upon how the point in the middle is interpreted, there are either two line segments crossing one another, or four line segments meeting at a point. We rule out such behavior. All polygons must have the property that each line segment intersects exactly two others, one at each endpoint.

The figure below is not a polygon because two of its line segments meet only one other line segment.

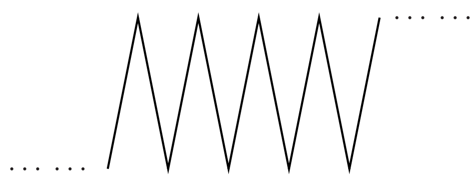


The following shape, although rather unusual in appearance, does satisfy the property that each line segment intersects exactly two others, one at each endpoint.



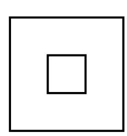
We will see momentarily that it does satisfy all the criteria for a polygon.

The following shape (the dots mean that the pattern continues indefinitely) has the property that each line segment intersects exactly two others, one at each endpoint, but it has infinitely many sides.

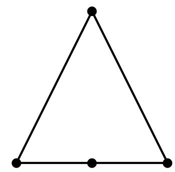


This is not a desirable feature for a polygon, so we add the condition that a polygon is made up of a *finite* collection of line segments.

Finally, consider this collection of line segments:



and this:



In the first example, the line segments make up two squares. This is not appropriate for a polygon, so we rule it out by requiring that no proper subcollection of line segments in a polygon satisfies all the same properties that we have discussed above. The second example has two adjacent line segments lying on the same line. We rule it out by not allowing a polygon to have two line segments that intersect at an endpoint to be collinear, that is, to lie on the same line.

Thus, we have the following definition of a polygon.
A *polygon* is a collection of a finite number of line segments such that each line segment intersects exactly two others, one at each endpoint, and such that no proper subcollection of line segments has the same property. In addition, two of the line segments which intersect at an endpoint cannot be collinear.

The line segments are called the *sides* or *edges* of the polygon, and the endpoints where the sides meet are called the *vertices* of the polygon. (The endpoints of a line segment are always considered to be distinct points.) A polygon having n sides and, consequently, n vertices, is called an n-gon. The *perimeter* of a polygon is the sum of the lengths of its sides.
Note. Some textbooks do not define the word "polygon" precisely and others have a definition of "polygon" with different requirements.

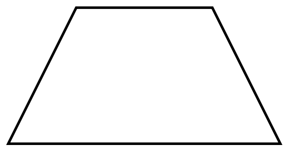
Recall that a triangle, i.e., a 3-gon, is *isosceles* if two sides have the same length, *equilateral* if all sides have the same length, *right* if it has a right, that is a $90°$, angle, and *scalene* if it is neither isosceles nor right. (Some texts say that a triangle is scalene if no two of its sides have equal length. In the classroom, right triangles receive special attention, so we prefer that the classes of right triangles and scalene triangles be mutually disjoint.)

In the definitions of isosceles and equilateral triangles, care has been taken to state that two, or three, sides have equal length. The abreviated form in which the sides themselves are stated to be equal is convenient and is often used. This shorthand will sometimes be used in these seminars. A more accurate statement is to say that the sides are "congruent" which implies that the sides have the same length.

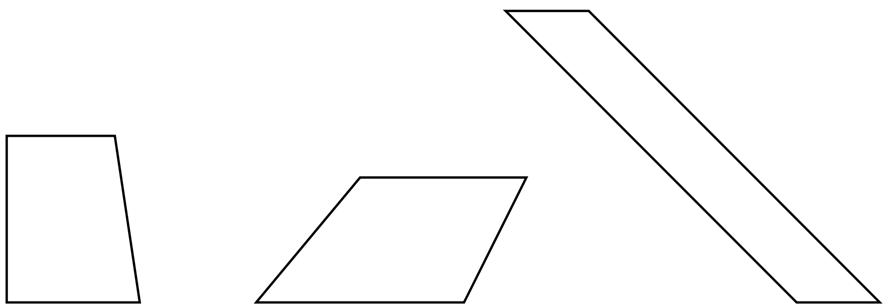
A polygon with four sides is a *quadrilateral*. A quadrilateral with four equal angles is a *rectangle*. (We adopt the convention of substituting the phrase "equal angles" for the phrase " congruent angles.") Note that the four equal angles of a rectangle are right angles. A *square* is a rectangle with four sides of equal length. A *parallelogram* is a 4-gon with opposite sides parallel. (Properties of parallel lines imply that the opposite sides of a parallelogram have equal length, see Section 6.) A *rhombus* is a parallelogram with four sides of equal length. A *trapezoid* is a 4-gon with at least one pair of parallel opposite sides.

Observe that the definitions above imply that a parallelogram is a special type of trapezoid. In other words, a parallelogram is a trapezoid with two pairs of parallel opposite sides. Many textbooks require that a trapezoid have *exactly* one pair of parallel opposite sides, but this definition is too restrictive for many purposes.

The picture that comes to mind when the word "trapezoid" is used is this



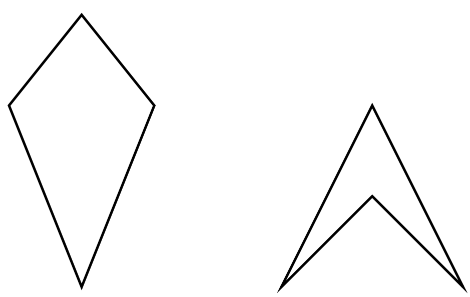
But, there are many types of trapezoids, for example:



All three figures are trapezoids; only the third is a parallelogram.

It is interesting to observe that, while a triangle with equal sides has equal angles, a quadrilateral with equal sides need not have equal angles. For example, a rhombus that is not a square does not have equal angles.

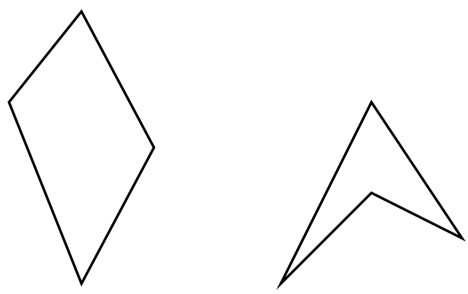
What distinguishes the two quadrilaterals below?



Notice that all pairs of vertices of the 4-gon on the left can be connected by line segments that lie inside (or on) the 4-gon, whereas this is not true about the 4-gon on the right. We can look for this same property in any n-gon, so we say that a polygon is *convex* if the line segment joining each pair of vertices lies within the polygon. (This definition of convex polygon implies that a line segment joining *any* two points of a polygon lies within the polygon.) A polygon is *nonconvex* if it does not have this property, that is, if there is a line segment joining two vertices that does not lie wholly in the polygon.

It is clear from the definitions that every triangle is convex. Rectangles, parallelograms and trapezoids are convex polygons. The 4-gon on the left in the diagram above is called a kite. A *kite* is a convex quadrilateral with two distinct pairs of equal adjacent sides. The 4-gon on the right above is called a dart. A *dart* is a nonconvex quadrilateral with two distinct pairs of equal adjacent sides. The unusual polygon drawn earlier is also nonconvex.

Note that not every quadrilateral has a "name." However, every quadrilateral is either convex or nonconvex. "Nameless" convex quadrilaterals and "nameless" nonconvex quadrilaterals exist. Small adjustments to the quadrilaterals pictured above yield the following examples of "nameless" quadrilaterals.



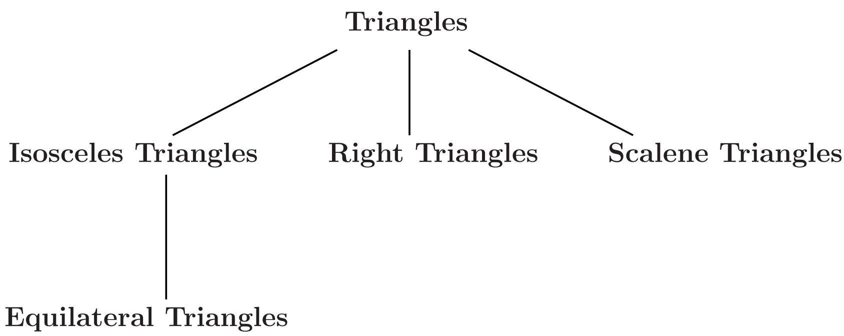
2. Classification of Triangles

When we teach mathematics, we start with particular, familiar examples, and move up to general, more inclusive concepts with greater breadth and depth. However, it is important to follow up by taking the return trip from the general back to the particular. This reinforces the ideas that were introduced.

An excellent way for your students to organize and to understand the concepts we have discussed so far today is to create a hierarchy of the various types of triangles, and a hierarchy of the various types of quadrilaterals.

First we will classify triangles. Every triangle is one of the three types: isosceles, right and scalene. To establish a hierarchy of triangles, we must distinguish, among all triangles, the ones that do not satisfy the defining properties of any of the others. Then we look for triangles that are special cases of the more general ones.

The Classification of Triangles



For example, by definition, scalene triangles do not satisfy the defining properties of any of the other three types. In other words, the answer to the question "Is a scalene triangle isosceles or right?" is, "No." Conversely, the answer to the question, "Is an isosceles or a right triangle scalene?" is also "No." It should be obvious that an equilateral triangle *must be* isosceles, whereas, a right triangle *can be* isosceles, but *need not be* isosceles.

The classification diagram drawn above displays these relationships.

When we read the diagram above from top down we pass from the general to the particular. The line from Triangles to Isosceles, for example, means that isosceles triangles satisfy the definition of triangle, and the line from Isosceles to Equilateral means that equilateral triangles satisfy the definition of isosceles triangle. Consequently, equilateral triangles are a subclass (or subtype) of the class of isosceles triangles and isosceles triangles are a subclass (or subtype) of the class of triangles. Observe that Isosceles Triangles and Right Triangles are not mutually exclusive subclasses. However, neither subclass includes the other. Our definition makes Right Triangles and Scalene Triangles mutually exclusive classes.

3. Classification of Quadrilaterals

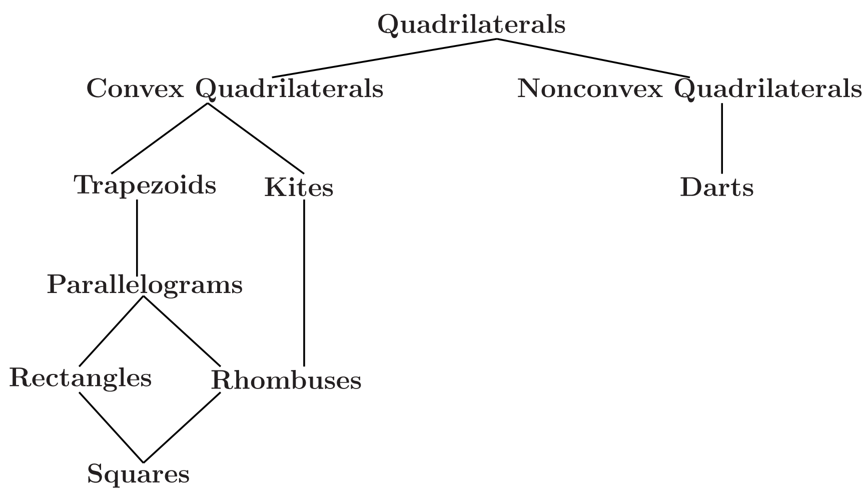
Whereas all triangles are convex, not all quadrilaterals are. We separate all quadrilaterals into two classes: convex and nonconvex nonconvex . The convex quadrilaterals are trapezoids, kites, parallelograms, rectangles, rhombuses, squares and convex quadrilaterals that are not one of the aforementioned familiar ones. Except for darts, we have no particular names for the nonconvex ones.

We classify convex quadrilaterals by naming the most general types first. This means we must distinguish, among all of the convex quadrilaterals, the ones that do not satisfy the defining properties of any of the other convex quadrilaterals. Then we look for convex quadrilaterals that are special cases of the most general convex ones.

Recall that with our nonrestrictive definition of trapezoid, a parallelogram is a trapezoid. After some thought, we discern that there are two general classes of "named" convex quadrilaterals: trapezoids and kites. Neither of these satisfies the defining properties of the other, that is, a trapezoid is not a kite and a kite is not a trapezoid.

Which convex quadrilaterals satisfy the definition of a trapezoid? All parallelograms do. Moreover, rectangles, rhombuses and squares not only satisfy the definition of trapezoid, they also satisfy the definition of a parallelogram. Which quadrilaterals satisfy the definition of a kite? Rhombuses and squares do. Moreover, a square is a particular type of rectangle and a particular type of rhombus. We organize all of this information in the classification diagram below.

The Classification of Quadrilaterals



The diagram can be read starting at any line. For example, starting with Parallelograms, the line up to Trapezoids means that every parallelogram satisfies the definition of a trapezoid. The lines down from Parallelograms to Rectangles and Rhombuses mean that every rectangle and every rhombus is a parallelogram.

Remember many convex quadrilaterals are neither trapezoids nor kites. They can be classified only as "convex quadrilaterals." Similarly, there are many nonconvex quadrilaterals that are not darts. They can be classified only as "nonconvex quadrilaterals."

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Classroom Activity.
Directions for the Students:

(i) Carefully draw and cut out an example of each of the named types of convex quadrilaterals. Be sure your examples distinguish one type from another.
(ii) Arrange the shapes to form a classification diagram for convex quadrilaterals.

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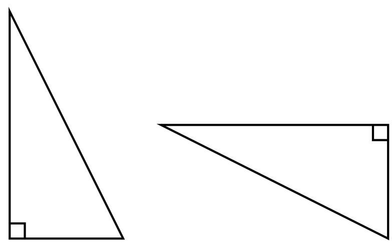
4. Congruence

The following two triangles are congruent.



In everyday language we think of congruence in terms of corresponding sides and angles. We also know that congruence of two triangles means that one triangle can be picked up and placed to fit perfectly on top of the other. Let's think more about that. How do we get, say, the left hand triangle, in the picture above, to fit on top of the right hand triangle? We have to rotate it, reflect (or flip) it and, finally, translate it.

The notion of congruence can apply to geometric figures other than triangles. The meaning is the same: two geometric figures are congruent if one figure can be picked up and placed on top of the other in a perfect match. Sometimes we use the word "equal" when we really mean "congruent." We might say, for example, that two line segments of the same length are "equal" when, in fact, they are distinct, so the correct term is "congruent." At other times, we might say that one figure is a "copy" of another. This is a descriptive way of expressing that we have a second figure that is congruent to the first. This informal language is perfectly fine as long as there is no possibility of confusion.

In more mathematical language, we say that two geometric figures are *congruent* if one of the figures can be transformed into the other by a sequence of transformations of the plane called *isometries* or *rigid motions*. A transformation of the plane is an *isometry* or *rigid motion* if it preserves length and angles. Isometries are precisely the transformations we have been discussing: rotations, reflections, translations and combinations of these.

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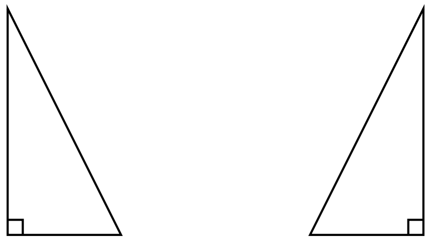
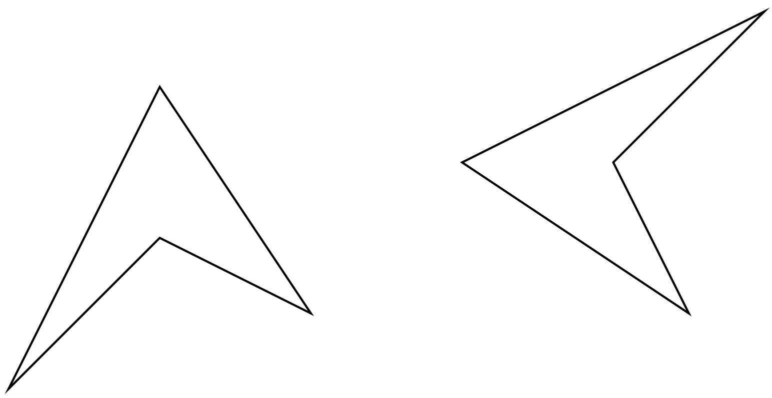
Classroom Activity.
Directions for the students:

(i) Consider the following two congruent figures:



Find what rotations, reflections (flips) and/or translations are needed to make the second figure fit precisely on top of the first.

(ii) Consider the following two congruent figures:



Find what rotations, reflections (flips) and/or translations are needed to make the second figure fit precisely on top of the first.

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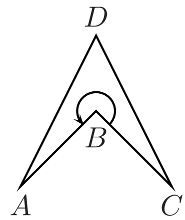
There are many sequences of isometries that will accomplish each task. For example, for the right triangles pictured, one reflection (flip) around the vertical leg and one horizontal translation shows that the second right triangle is congruent to the first.

5. Angles: A Review and Some Remarks

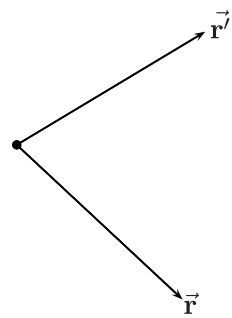
(When convenient, we abbreviate the terminology "An angle has measure x degrees" and write "An angle has x degrees.")

We begin with a review of standard angle vocabulary. A *straight* angle has exactly 180°. An *acute* angle has less than 90°. An *obtuse* angle has greater than 90° and less than 180°. Also, we have *complementary* angles, two angles whose sum is 90°, and *supplementary* angles, two angles whose sum is a straight angle.

What about angles having more than 180°? These are the angles that are larger than a straight angle. For example, $\angle ABC$ inside the dart drawn below, has more than 180°.

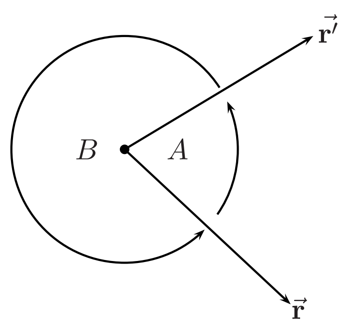


For the most part, these angles are ignored in textbooks. Frequently, the definition of an angle given is that an angle is a pair of rays that have the same endpoint.



But then how do we measure such an angle? It appears that we are supposed to understand that angles never measure more than 180°. If so, then discussion of the interior angles of a nonconvex quadrilateral is impossible. (See Seminar 2 for more on the interior angles of a polygon.)

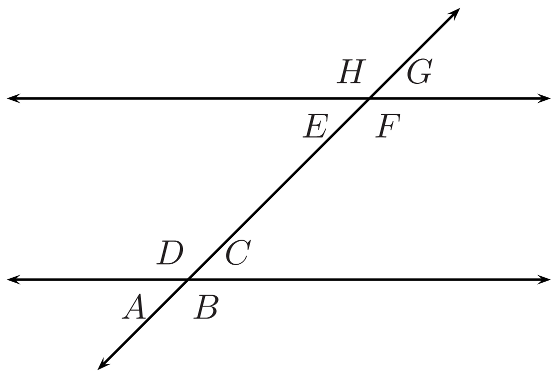
A better definition is to say that two distinct rays $\vec{\mathbf{r}}$, $\vec{\mathbf{r}'}$ with the same endpoint separate the plane into two regions and each one of these regions is called an angle determined by the rays.



There is angle A from $\vec{\mathbf{r}}$ counterclockwise to $\vec{\mathbf{r}'}$, and there is angle B from $\vec{\mathbf{r}'}$ counterclockwise to $\vec{\mathbf{r}}$. Consequently, two rays with the same endpoint determine two angles. The context must make clear which one is meant.

6. Angles and Parallel Lines

As you know, two parallel lines and the angles associated to them when they are cut by a transversal, that is, by a line that intersects both, appear frequently in geometry. Here, we recall the facts about the different pairs of angles that arise when two lines (not necessarily parallel) are cut by a transversal. The lines may meet somewhere off the page.



The angles A and C are called *vertical angles*. This means that the sides of one angle are opposite rays to the sides of the other angle. Other pairs of vertical angles are B, D and E,G and F, H. The angles C, G are *corresponding angles*. Other pairs of corresponding angles are A, E and B, F and D, H. The angles E and C, and F and D are called *alternate interior angles*. The angles D and E are called *interior angles on the same side of the transversal*. The angles C and F are another such pair.

If you need to show that two lines or line segments are parallel, here are some basic facts that might help you do it. Suppose that we have two lines (they might be two opposite sides of a quadrilateral) and a third line that is a transversal (it might be one of the sides adjacent to the two opposite sides).

(i) If two corresponding angles are equal, then the lines are parallel.
(ii) If two alternate interior angles are equal, then the lines are parallel.
(iii) If two interior angles on the same side of the transversal are supplementary, then the lines are parallel.

Suppose, on the other hand, that we have two parallel lines or line segments, and suppose that the lines are cut by a transversal. Then

(i) All pairs of corresponding angles are equal.
(ii) All pairs of alternate interior angles are equal.
(iii) All pairs of interior angles on the same side of the transversal are supplementary.

7. Homework

Exercise 1.1. *Add five more words to the list of words associated with geometry in the Preface. Draw a picture illustrating each term on the expanded list. Do not write anything more. Geometry is a subject of diagrams and pictures.*

Exercise 1.2. *Without looking at the diagrams in the text, draw a classification diagram for triangles and one for quadrilaterals. Illustrate the diagrams by drawing a picture of each type of polygon in the classification.*

Exercise 1.3. *Draw a polygon with twenty nine sides.*

Exercise 1.4. *Let Q be a convex quadrilateral. Let P be the quadrilateral formed by joining the midpoints of adjacent sides of Q. Show that P is a parallelogram.*

Exercise 1.5. *Look in your math textbooks and determine how the authors define angles. Are angles with measure greater than 180° allowed?*