Introduction

The Moscow Mathematical Olympiad, now in its 76th year, has been challenging high-school students from Moscow and elsewhere with stimulating, original problems of varying degrees of difficulty. Its problems are nonstandard; solving them takes wit, thinking outside the box, and sometimes hours of contemplation. Some are within the reach of most mathematically competent high-school students, while others are difficult even for a math professor.

Many mathematically inclined students have found that tackling these problems, or even just reading their solutions, is a great way to develop mathematical insight.

In 2006 the Moscow Center for Continuous Mathematical Education, or MCCME, published a collection of 13 years of problems from the Moscow Mathematical Olympiad (MMO), covering the period from 1993 to 2005, and providing for each an answer (and sometimes a hint) as well as one or more detailed solutions. This work, described in more detail below, has now been translated into English, and will appear in two volumes, of which this is the first, spanning the years 1993–1999.

Why Olympiad problems?

An important feature of Olympiad problems is that they don't require any knowledge beyond the school curriculum. Of course, this is only approximately true, because certain usually extracurricular methods such as mathematical induction have been accepted by the Olympiad community as legal.

But if the problems do not routinely require special knowledge, what is the difference between an Olympiad and a puzzle competition? We believe that good Olympiad problems are deeply related to "real mathematics", and illustrate fundamental mathematical ideas. MMO problems are known for such connections; for example, Problems 93.A.2 and 96.D.4 are related to the ancient question of writing a number as a sum of two perfect squares, while Problem 99.C.5 is related to what's known as contracting maps. Other problems where such connections are striking include 93.C.4, 95.C.6, 97.D.3, 00.C.3,[1] 01.C.5, 02.D.6, 03.C.6, 04.B.3, 04.C.3, 05.A.5, 05.C.6. In this book

[1]Problem references starting in 0 can be found in the companion volume, *Moscow Mathematical Olympiads, 2000–2005.*

we made a special effort to show connections with real mathematics in the comments.

For many Russian mathematicians, including some of the present authors, the compilations [561] and [562][2] of problems from the Moscow Mathematical Olympiad were among the first mathematical books they read. We hope that young readers of this book will likewise be touched by the beauty of our favorite science.

What is the Moscow Mathematical Olympiad?

The MMO is a contest for high-school students, that is, students in the last four years of the Russian school system, roughly corresponding to the 9th to 12th grades in the United States. Every year, at the end of February or the beginning of March, two to three thousand students come to Moscow State University to compete in solving mathematical problems. Most range from 13 to 17 years of age. There is one test for each school year (see next section). In fact, students are allowed to compete at higher levels (that is, with older students), but this is not encouraged. The Olympiad is open: any Moscow or non-Moscow high-school student can participate.

The Olympiad consists of a single round. Students are generally given six problems and five hours to solve them. Then the papers are carefully graded and the prizes, in the form of mathematical books, are awarded at the closing ceremony, which usually takes place two weeks after the Olympiad. There are four levels of awards: first-, second- and third-degree diplomas, and honorable mentions. First-degree diplomas are usually awarded to between one and five students from each level; honorable mentions are given to fifty or more students from each level. The problems are difficult: an honorable mention is normally awarded for solving three problems, but sometimes even two are enough.

About this book

In the Russian edition, the statement of each problem was taken verbatim from the corresponding Olympiad. In a translation, literal equivalence is not possible in any case, and the translation editor, Silvio Levy, felt free to depart from the phrasing of the original when he thought this would help spare the reader some puzzlement. Of the 160 or so problems in this volume, a handful are outright adaptations, though respecting the mathematical content, while many others were translated freely rather than literally.

The numbering of the problems was also changed, the Russian class level (8, 9, 10, 11) being replaced by a letter (A, B, C, D), to avoid the implication that level-11 problems, say, should necessarily be accessible to 11th grade students in the United States. In fact a better fit for level-D

[2]Three-digit numbers in brackets refer to the bibliography on page 213.

problems would be 12th graders, but the equivalence is not exact, especially since the curriculum in American high schools is far from standardized.

Difficulty level. The MMO organizers try to order the problems for a given level and year by difficulty. Ideally, the first problem is the easiest and ends up being solved by a quarter to half of all the participants, while the last problem is solved by only a few. Problems that are too difficult for their positions (in our opinion) are given a star in this book. For example, we think that 97.C.1 is too difficult as a first problem, but as a second, we would not give it a star. Of course, that Problem 97.C.6 has no star does not imply that it is easier than 97.C.1. Whenever statistics from the Olympiads were available, we used them as a guide in assigning stars.

Solutions. We always tried to give a solution formulated from within the context of the (Russian) high-school curriculum; in most cases we managed to do this. For some problems we give a second solution, one that requires facts or ideas beyond the curriculum. In some cases, the second solution rests on the first, in the sense that we assume that the reader who has studied the first solution does not mind learning a more elegant one.

Reference facts. This part contains the concepts and theorems used most frequently in the solutions. We tried not to include those generally covered by high-school textbooks, except in cases where we deemed the curriculum inadequate. For every fact, we give a cross-referenced list of problems. Should you encounter a fact you are not familiar with while reading a solution, try to solve other problems where this fact is also used.

Hints. We've tried to offer hints that don't give away the answer, so don't expect a problem to become trivial after reading a hint. If you still cannot solve the problem, read the answer; if even this does not help, read the beginning of the solution.

Acknowledgements

Every Olympiad is the result of the work of many problem authors; a list of authors for this volume is given at the end. We also want to mention the enormous work of those who selected the problems, compiled the tests, proctored the tests, and graded the papers. Finally, Olympiads have no raison d'être without their participants — motivated high school students. We note with great pleasure that many of the students who solve Olympiad problems as contestants go on to become professional mathematicians.

After each Olympiad, a booklet with solutions is printed, and handed out to participants at the closing ceremony. We have used these booklets as the basis for our book. We have corrected mistakes (and added new ones, no doubt), improved solutions, provided greater detail and illustrations, added second solutions, and so on. We are thankful to all those who participated in writing these booklets.

Sometimes it is difficult for a writer to know whether a given solution will be intelligible to a student. We are grateful to A. D. Blinkov, A. M. Fedorova,

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Most of the illustrations were drawn by Nadezhda Shikhova and Vadim Radionov; the remainder by one of us (Fedorov) and the translation editor.