Problems

1993 Olympiad

Level A

Problem 1. Denote by $S(x)$ the sum of the digits of a positive integer x. Solve:

(a) $x + S(x) + S(S(x)) = 1993.$

(b) $x + S(x) + S(S(x)) + S(S(S(x))) = 1993.$

Problem 2*. Suppose n is the sum of the squares of three positive integers. Prove that n^2 is also the sum of the squares of three positive integers.

Problem 3. A red and a blue poker chip are stacked, the red one on top. Suppose one can carry out only the following operations: (a) adding two chips of the same color to the stack together, in any position; and (b) removing any two neighboring chips of the same color. After finitely many operations, is it possible to end up with only two chips left, the blue one on top of the red one?

Problem 4. At the court of Tsar Gorokh, the royal astrologer built a clock remarkably similar to modern (analog) ones, with hands for hours, minutes, and seconds, all moving smoothly around the same point. He calls a moment of time *lucky* if the three hands of his clock, counting clockwise from the hour hand, appear in the order hours/minutes/seconds, and *unlucky* if they appear in the order hours/seconds/minutes. Is the amount of lucky time in a 24-hour day more or less than the amount of unlucky time?

Remark. Tsar Gorokh (King Pea) is a character from Russian folklore. “In the time of Tsar Gorokh” is a Russian idiom meaning “a very long time ago”.

Problem 5. Prove or disprove: There is a finite string of letters of the alphabet such that there are no two identical adjacent substrings, yet a pair of identical adjacent substrings appears as soon as one adds any letter of the alphabet at the beginning or at the end of the string.

Problem 6. A circle centered at D passes through points A, B, and the excenter O of the triangle ABC relative to side BC (that is, O is the center of the circle tangent to BC and to the extensions of sides AB and AC). Prove that A, B, C, and D lie on a circle.

Level B

Problem 1. For two distinct points A and B in the plane, find the locus of points C such that the triangle ABC is acute and the value of its angle at A is intermediate among the triangle's angles (meaning that $\angle B \le \angle A \le \angle C$ or $\angle C \le \angle A \le \angle B$).

Problem 2. Let $x_1 = 4,\ x_2 = 6$, and define x_n for $n \ge 3$ to be the least nonprime greater than $2x_{n-1} - x_{n-2}$. Find x_{1000}.

Problem 3. A paper triangle with angles of 20°, 20°, and 140° is cut along one of its bisectors into two triangles; one of these triangles is also cut along one of its bisectors, and so on. Can we obtain a triangle similar to the initial one after several cuts?

Problem 4. In Pete's class, there are 28 students besides him. Each of the 28 has a different number of friends in the class. How many friends does Pete have in this class?

Problem 5. To every pair of numbers x and y we assign a number $x * y$. Find $1993 * 1935$ if it is known that

$$x * x = 0 \quad \text{and} \quad x * (y * z) = (x * y) + z \quad \text{for any } x, y, z.$$

Problem 6. Given a convex quadrilateral $ABMC$ with $\angle BAM = 30°$, $\angle ACM = 150°$, and $AB = BC$, prove that AM is the bisector of $\angle BMC$.

Level C

Problem 1. In the decimal representation of two numbers A and B, the minimal periods have lengths 6 and 12, respectively. What are the possibilities for the length of the minimal period of $A + B$?

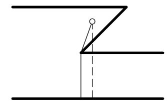
Problem 2. The grandfather of Baron von Münchhausen built a castle with a square floor plan. He divided the castle into 9 equal square areas, and placed the arsenal in the center square. The Baron's father divided each of the remaining 8 areas into 9 equal square halls and built a greenhouse in each central hall. The Baron himself divided each of the 64 empty halls into 9 equal square rooms and placed a swimming pool in each of the central rooms. He then furnished the other rooms lavishly and connected each pair of adjacent furnished rooms by a door, locking all other doors.

The Baron boasts that he can tour all his furnished rooms, visiting each exactly once and returning to the starting point. Can this be true?

Problem 3. From any point on either bank of a river one can reach the other bank by swimming a distance of no more than 1 km.

(a) Is it always possible to pilot a boat along the whole length of the river while remaining within 700 m of both banks?
(b) * Same question with 800 m.

Remark. Both the answer and the degree of difficulty of the problem depend on what additional assumptions are made. Naturally the boat is to be considered a point. The original problem had a note saying so, and also that "the river joins two round lakes, each 10 km in radius, and the river banks consist of straight line segments and arcs of circle." But in spite of this precision, ambiguities remain. Can there be islands in the river? And does "within 700 m" refer to the straight-line distance or the swimming distance? (See figure.) *Warning:* Part (b) is surprisingly difficult, unless islands are allowed, in which case both parts are easy.



Problem 4. Given real numbers a and b, define $p_n = [2\{an+b\}]$, where $\{x\}$ denotes the fractional part of x and $[x]$ the integer part.

(a) Can all possible quadruples of 0s and 1s occur as substrings of the sequence $p_0, p_1, p_2, \ldots$, if we are allowed to vary a and b?
(b) Can all possible quintuples of 0s and 1s occur?

Problem 5. In a botanical classifier, a plant is identified by 100 features. Each feature can either be present or absent. A classifier is considered to be *good* if any two plants have less than half of their features in common. Prove that a good classifier cannot describe more than 50 plants.

Problem 6. On the side AB of a triangle ABC, a square is constructed outwards; let its center be O. Let M and N be the midpoints of AC and BC, and let the lengths of these sides be a and b. Find the maximum of the sum $OM + ON$ as the angle ACB varies.

Level D

Problem 1. Knowing that $\tan\alpha + \tan\beta = p$ and $\cot\alpha + \cot\beta = q$, find $\tan(\alpha+\beta)$.

Problem 2. The unit square is divided into finitely many smaller squares, not necessarily of the same size. Consider the small squares that overlap (possibly at a corner) with the main diagonal. Is it possible for the sum of their perimeters to exceed 1993?

Problem 3. We are given n points in the plane, no three of which lie on a line. Through each pair of points a line is drawn. What is the least possible number of pairwise nonparallel lines among these lines?

Problem 4. We start with a number of boxes, each with some marbles in them. At each step, we select a number k and divide the marbles in each box into groups of size k with a remainder of less than k; we then remove all but one marble from each group, leaving the remainders intact.

Is it possible to ensure that in 5 steps each box is left with a single marble, if initially each box has at most (a) 460 marbles, (b) 461 marbles?

Problem 5. (a) It is known that the domain of a function f is the segment $[-1, 1]$, and $f(f(x)) = -x$ for all x; also, the graph of f is the union of finitely many points and straight line segments (with or without endpoints). Draw a possible graph for f.

(b) Is it possible to draw the graph of f if the domain of f is $(-1, 1)$? the whole real line?

Problem 6*. A fly lives inside a regular tetrahedron with edge a. What is the shortest length of a flight the fly could make to visit every face and return to the initial spot?

Hints

1993 Olympiad

Level A

1. (a) Use divisibility by 3. (b) Estimate x from below and find the remainder after division of x by 9.
2. Rewrite the expression $(a^2 + b^2 + c^2)^2$ in a different form.
3. Consider all pairs of chips with a red chip above a blue chip.
4. Favorable and unfavorable times are interchanged by reflection.
5. Use induction on the number of letters in the alphabet.
6. Prove that $\angle ADB = \angle ACB$.

Level B

2. Write down $x_n - x_{n-1}$ for n small.
3. From what triangles can a triangle with the initial angles be obtained?
4. Consider the most friendly and the least friendly of Pete's classmates.
6. Consider the points symmetric to B with respect to AM.

Level C

1. The digits of $A + B$ must repeat in blocks of 12.
4. Draw the points $\{an + b\}$ on the circle of unit circumference.
5. Estimate the total number of distinctions between all possible pairs of plants with respect to all features.

Level D

4. Let there be 1, 2, ..., n stones in the boxes. Express the maximal number of stones after one move in terms of n and k. Investigate for which k this number is minimal.
5. The graph of such a function would be left unchanged by a 90° rotation around the origin.
6. Let P and Q be the midpoints of the sides KL and MN of a space quadrilateral $KLMN$. Then

$$PQ \leq \tfrac{1}{2}(KN + LM).$$

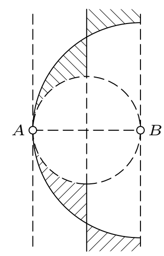
Answers

1993 Olympiad

Level A

1. (a) No solutions. (b) $x = 1963$. **3.** No. **4.** The amounts of lucky and unlucky time are the same. **5.** There is always such a string.

Level B

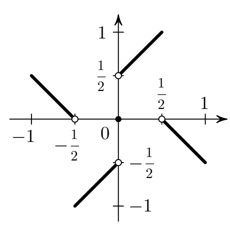


1. The locus is given by the shaded domains and solid curves in the figure. (The points on the dotted curves do not belong to the locus.) **2.** $x_{1000} = \frac{1}{2} \cdot 1000 \cdot 1003 = 501500$. **3.** No. **4.** Pete has 14 friends. **5.** 58.

Level C

1. 4 or 12. **2.** Yes. **3.** 700 m is not always possible, but 800 m is (if islands are not allowed). Here we're considering the swimming distance rather than the straight-line distance. **4.** (a) Yes. (b) No. **6.** The maximum equals $\frac{1}{2}(1+\sqrt{2})(a+b)$ and is attained when $\angle ACB = 135°$.

Level D



1. $pq/(q-p)$, unless $p = q$, in which case the answer is either 0 (if $p = 0$) or undefined (if $p \neq 0$). **2.** Yes. **3.** n lines if $n > 2$; one line if $n = 2$. **4.** (a) Yes. (b) No. **5.** (a) One solution is shown in the figure. (b) No. **6.** $\frac{4}{\sqrt{10}}a$.

Solutions

1993 Olympiad

Level A

Problem 1. (a) Since x, $S(x)$ and $S(S(x))$ have the same remainder upon division by 3 (by Fact 7), the sum $x+S(x)+S(S(x))$ is divisible by 3. Since 1993 is not divisible by 3, there is no solution.

(b) Clearly $x < 1993$. It is easy to see that 1899, 1989 and 999 have the largest sum of digits among the numbers from 1 to 1993. Thus $S(x) \leq 27$. Further, $S(S(x)) \leq S(19) = 10$ and $S(S(S(x))) \leq 9$. The equation to be solved implies that

$$x = 1993 - S(x) - S(S(x)) - S(S(S(x))) \geq 1993 - 27 - 10 - 9 = 1947.$$

As in part (a), we know that x, $S(x)$, $S(S(x))$ and $S(S(S(x)))$ leave the same remainders when divided by 9; let's call this remainder r. Then $4r$ has remainder 4 when divided by 9 (because that's the remainder of 1993). In other words, $4r-4$ is divisible by 9, which implies that $r-1$ is divisible by 9 (since 4 and 9 are coprime; see Fact 9). So $r=1$.

Now list the numbers from 1947 through 1993 that have remainder 1 upon division by 9: they are 1954, 1963, 1972, 1981 and 1990. We can verify directly that only 1963 satisfies the equation.

Problem 2. Let $n = a^2+b^2+c^2$. Expand the square and rearrange:

$$\begin{aligned}(a^2+b^2+c^2)^2 &= a^4+b^4+c^4+2a^2b^2+2a^2c^2+2b^2c^2\\ &= (a^4+b^4+c^4+2a^2b^2-2a^2c^2-2b^2c^2)+(2ac)^2+(2bc)^2\\ &= (a^2+b^2-c^2)^2+(2ac)^2+(2bc)^2.\end{aligned}$$

We can assume that $a \geq b \geq c > 0$, so $a^2+b^2-c^2 > 0$. Thus we have expressed n^2 as the sum of squares of three positive integers.

Remarks. 1. Try to prove a similar statement for the sum of four and more squares.

2. For the sum of two squares the analogous statement is not true: $(1^2+1^2)^2 = 4$ cannot be represented as the sum of two squares of positive integers, although an analogous identity is true:

$$(a^2+b^2)^2 = (a^2-b^2)^2+(2ab)^2.$$

For more on sums of two squares, see the solution to Problem 96.D.4 on page 134 and Remark 2 thereto.

3. A famous theorem of Lagrange says that any positive integer can be represented as the sum of four integer squares. (An *integer square*, or simply a *square*, means the same as the square of an integer. This includes 0 as well as the squares of positive integers.)
4. The number 7 is not a sum of three squares. It turns out that a positive integer is *not* representable as a sum of three squares if and only if it is of the form $(8k+7)\cdot 4^m$.
5. Suppose that instead of specific numbers we consider sums of squares of arbitrary variables. An identity going back at least to the third-century Greek mathematician Diophantus of Alexandria says that

$$(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2,$$

which of course can also be written as

$$(a_1^2+a_2^2)(b_1^2+b_2^2) = (a_1b_1+a_2b_2) + (a_1b_2-a_2b_1)^2.$$

This elegant equality was extended by Euler in the eighteenth century to the case of four pairs of variables:

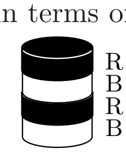
$$\begin{aligned}(a_1^2+a_2^2+a_3^2+a_4^2)&(b_1^2+b_2^2+b_3^2+b_4^2)\\ &= (a_1b_1-a_2b_2-a_3b_3-a_4b_4)^2+(a_1b_2+a_2b_1+a_3b_4-a_4b_3)^2\\ &\quad +(a_1b_3+a_3b_1+a_4b_2-a_2b_4)^2+(a_1b_4+a_4b_1+a_2b_3-a_3b_2)^2.\end{aligned}$$

A similar identity also exists representing

$$(a_1^2+a_2^2+\cdots+a_k^2)(b_1^2+b_2^2+\cdots+b_k^2) \tag{1}$$

as a sum of squares of polynomials when $k=8$. However, *for no other* k — apart from the already discussed $k=2,4,8$ and the trivial case $k=1$ — is the product (1) writable as a sum of squares of polynomials! The proof of this fact is far from elementary; to intrigue the reader, we mention only that the identity for $k=2$ is related to complex numbers, for $k=4$ to quaternions, and for $k=8$ to Cayley numbers, also called octonions. More about this in the book [413].

Problem 3. *First solution.* Consider *all* pairs of chips in the stack, not necessarily adjacent. There are four possibilities for such pairs, in terms of the ordering of colors: RR, RB (meaning a red chip above a blue chip), BR and BB. For instance, in the stack RBRB of the figure, there are three RB pairs (first and second chips from the top, first and last, third and last).



The reader can check that the parity (see Fact 23) of the number of RB pairs cannot change under our operations. For example, suppose we insert two red chips at a spot that has k blue chips below it. It is easy to see that this adds exactly $2k$ new RB pairs, so the parity remains the same. The other operations — insertion of two blue chips and removal of two blue or two red chips — can be analyzed similarly.

Now, in the initial position, there is exactly one RB pair. In the desired final position, there are none. Since 1 and 0 have different parities, it's not possible to reach one state from the other.

Remark. The parity of the number of RB pairs in this solution is an *invariant* (see Fact 2), since it remains unchanged no matter what happens to the stack, under the rules of the problem. One could alternatively consider the number of red chips below which there is an odd number of blue chips; the parity of this number is also invariant. (Check it!)

The next solution also relies on an invariant, but of a different type: a transformation of the real line into itself. (A *transformation* is a map that is one-to-one and onto, and therefore has an inverse.)

Recall that two maps from a set to itself can be *composed*, that is, applied successively, producing a third map. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are maps of the real line, the composition fg (also written $f \circ g$) is the map obtained by applying g, then f:

$$(fg)(x) = f(g(x)).$$

The composition of any map f with the *identity* (the map Id such that $\mathrm{Id}(x) = x$ for all x) is of course f again. The composition of several maps depends on the order in which they are applied, but not on the grouping: in symbols, $(fg)h = f(gh)$, because both of these maps take x to $f(g(h(x)))$.

Second solution. Consider two transformations of the real line, r and b, defined by $r(x) = 1 - x$ (this is a *reflection* in the point $\frac{1}{2}$ of the line) and $b(x) = -1 - x$ (a reflection in the point $-\frac{1}{2}$).

Obviously, composing r with r brings each point of the line back to itself, and so gives the identity map; in algebraic notation (see preceding remark), $rr = \mathrm{Id}$. Similarly, $bb = \mathrm{Id}$.

The composition of two reflections of the line is, in general, a translation. For instance, applying b, then r, has the overall effect of a translation to the right by 2:

$$(rb)(x) = r(b(x)) = 1 - (-1 - x) = x + 2. \tag{1}$$

Now let's associate to red chips R the reflection r, and to blue chips B the reflection b. For any stack of chips, imagine reading off the colors from the bottom of the stack up and applying consecutively the reflection corresponding to each color. The overall effect of all these reflections r and b, in the given order, is again a map of the real line.

The initial stack in the problem, RB, corresponds to applying b then r; this, according to equation (1), is the translation by 2. But if we consider the reverse stack BR, the composition gives a different result altogether:

$$b(r(x)) = -1 - (1 - x) = x - 2.$$

The composition of transformations is not, in general, commutative!

Now the key observation is this: two adjacent chips of the same color *cancel each other out* from the point of view of composition: their combined effect, as we have seen, is the identity. This means that inserting or removing two adjacent chips of the same color has no effect on the overall map represented by the stack of chips; for example, $rb = bbrb = brrbrb$. Therefore no sequence of such insertions of removals can lead from the stack RB to the stack BR, as they represent different overall maps.

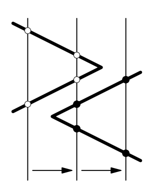
Remarks. 1. (Follow-up on the second solution, for more advanced readers familiar with group theory language.) The set of compositions is the *group of transformations* of the real line *generated* by r and b, and these generators satisfy the *relations* $r^2 = 1$ and $b^2 = 1$.

One can also define an *abstract group* with generators r and b — these letters now being regarded just as symbols — subject to the relations $r^2 = 1$ and $b^2 = 1$, meaning that whenever bb or rr appear in a product of generators, they can be erased. By interpreting multiplication as composition we obtain a map from the abstract group to the group of transformations; it can be checked that this is an isomorphism (that is, the group of transformations has no additional relations beyond those in the abstract group). Mathematicians call such an isomorphism a *faithful representation* of the abstract group in $\mathbb{R}$. This abstract group can be shown to be isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}/2$, the *semidirect product of* $\mathbb{Z}$ *and* $\mathbb{Z}/2$.

2. The problem we've solved (twice!) comes up in a proof of an amusing and geometrically intuitive fact: *Two continuous paths inside a square, each of which joins a pair of opposite vertices, must intersect.* We sketch the proof in the case where the paths are polygonal. (The reader acquainted with continuity might try to prove that this implies the general case, using two facts from topology: the image of a continuous path $[0, 1] \to \mathbb{R}^2$ is *compact*; and if two compact sets don't intersect, there is some minimum positive distance between points in one set and points in the other.)

   So let's consider two nonintersecting polygonal paths as described, one blue and one red, inside a square. We make the square's sides vertical and horizontal. Now move a vertical line continuously from the left edge of the square to the right edge, keeping track of the intersections of this line with the red and blue paths. The pattern of red and blue intersection points corresponds to the stack of chips in the problem. (Since the paths don't intersect we can assume that none of the polygonal segments is vertical: just jiggle a vertex a tiny bit if it is. Therefore the intersection of either path with a vertical line is a finite set.)

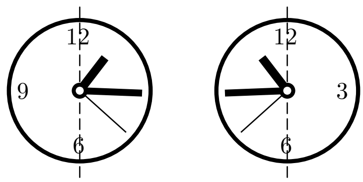
   The intersection points always appear and disappear in pairs of a single color, as in the problem. (See the diagram on the right for an appearing pair and a disappearing one.) Therefore if on the left edge of the square we have the red dot above the blue dot, we cannot have the inverse situation on the right edge, contradicting the assumption that the paths join opposite corners.



3. The geometric result just proved is closely related to the Jordan curve theorem, which states that *every closed plane curve without self-intersections divides the plane into two parts*, an inside and an outside. This intuitively clear statement is difficult to prove, since the curve may have no smooth pieces. But in the case of polygonal closed curves an argument very similar to the one above provides a proof.

Problem 4. The key idea is that the mirror image of a lucky pattern of hands is unlucky, and vice versa.

Consider the position of the hands at two distinct times: T seconds before noon and T seconds after noon. The patterns formed by the hands of the clock at those moments are mirror images of each other, with respect to the axis of symmetry formed by a vertical diameter. (Why?) For example, here is a clock showing 1 hour, 15 minutes, and 22 seconds after noon (01:15:22 PM) and another showing the same amount of time before noon (10:44:38 AM):



Now, a moment's thought shows that reflection interchanges lucky and unlucky patterns of hands. Thus, to each lucky moment before noon there corresponds an unlucky moment after noon. An interval of lucky moments in the morning is matched by an interval of unlucky moments in the afternoon or evening, *and both intervals have the same length.* Similarly, an interval of unlucky moments before noon is matched by an equal interval of lucky moments after noon. Hence the total amount of lucky time in the day equals the total amount of unlucky time.

Remarks. 1. It's not enough to establish a one-to-one correspondence between lucky and unlucky moments; we need to consider the *length* of the intervals. This question is discussed further in Remark 2 after the solution to Problem 94.A.5, which deals with a similar situtation (see page 85).

2. It is also not enough to argue that each lucky pattern of hands has a mirror image that is an unlucky pattern of hands, because the mirror image might conceivably not correspond to an actual moment in time. Some combinations of hand positions simply cannot occur on a properly functioning clock. (Find examples!) So one must establish a correspondence between actual intervals of time.

Problem 5. Consider the sequence of strings

A, ABA, ABACABA, ABACABADABACABA, ...

each string being obtained from the previous one by writing it twice and inserting the first unused letter between the two copies.

When we run out of letters, the last string will provide an affirmative answer to the problem. We will prove this by complete induction (Fact 24). For convenience, we will denote the n-th string of the sequence by Z_n, so $Z_1 =$ A, $Z_2 =$ ABA, etc. We also denote the n-th letter of the alphabet by x_n, so $x_1 =$ A, $x_2 =$ B, ..., $x_{26} =$ Z. Then the sequence of strings is defined by the properties

$$Z_1 = x_1 \quad \text{and} \quad Z_{n+1} = Z_n x_{n+1} Z_n.$$

The "multiplication" on the right-hand side simply means that we are concatenating (writing one after the other) the string denoted by Z_n, the letter x_{n+1}, and again the string Z_n, is this order.

Obviously this process stops when we run out of letters, that is, after $n = 26$. We claim that *for any* $n \le 26$, (a) *the string* Z_n *doesn't have identical adjacent substrings*, but (b) *a pair of identical adjacent substrings appears as soon as one writes any one of the first* n *letters of the alphabet either at the beginning or at the end of* Z_n.

Base of the induction. For $n = 1$, the statement is obvious.

Induction step. Suppose (a) and (b) are true for $Z_1, \dots, Z_{n-1}$, and consider the n-th string, $Z_n = Z_{n-1}x_nZ_{n-1}$.

Suppose Z_n has two identical adjacent substrings. They cannot contain the central letter x_n, since there is only one copy of it. Therefore they lie both to the left or both to the right of the central letter; that is, they're identical adjacent substrings of Z_{n-1}. But such substrings cannot exist, by the induction assumption. This proves statement (a) for Z_n.

To prove (b), again we proceed by cases. Suppose we write after Z_n one of the first n letters of the alphabet. If the letter we wrote is the central letter x_n, the result is two copies of $Z_{n-1}x_n$. If we wrote any other letter x_k, the string now ends with $Z_{n-1}x_k$, with $k < n$. But the induction hypothesis says that $Z_{n-1}x_k$ contains two identical adjacent substrings somewhere. Either way, we've found the desired identical substrings. The argument also applies if we write a letter to the left of Z_n instead of to the right.

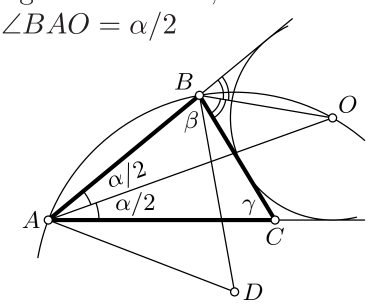
Having proved the induction, we now just take $n = 26$, so all the letters of the alphabet are allowed. (The word Z_n thus constructed has $2^{26} - 1$ letters, or more than 50 million!)

Remark. This construction is important in combinatorics and the theory of semigroups. Try to prove that in any *infinite* sequence of letters of the alphabet, there must be somewhere a string *in the pattern of* Z_n, for any n. By a string in the pattern of Z_n we mean one that is obtained from Z_n by replacing each letter x_i by a fixed nonempty string X_i; for instance "abracadabra" is a string in the pattern of Z_2 (take $X_1 =$ abra and $X_2 =$ cad).

Problem 6. The circle tangent to BC and to the extensions of sides AB and AC is an *excircle* of the triangle ABC. Its center O is the intersection of the bisectors of the angle A and the exterior angle at vertex B; see Fact 16.

Let α, β, γ be the angles of $\triangle ABC$. Then $\angle BAO = \alpha/2$ and $\angle CBO = (180^\circ - \beta)/2$, and

$$\angle ABO = \beta + \angle CBO = 90^\circ + \beta/2.$$



From $\triangle AOB$ we see that

$$\angle AOB = 180^\circ - \alpha/2 - (90^\circ + \beta/2)$$
$$= 90^\circ - \alpha/2 - \beta/2 = \gamma/2,$$

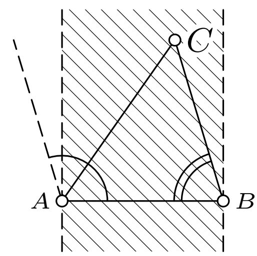
since $\alpha + \beta + \gamma = 180^\circ$. On the other hand, we have $\angle AOB = \frac{1}{2}\angle ADB$, because this is the angle inscribed in the circle centered at D. Hence $\angle ADB = \gamma$.

Thus, $\angle ADB = \angle ACB$. By the converse of the theorem on inscribed angles, the points A, B, C and D lie on one circle.

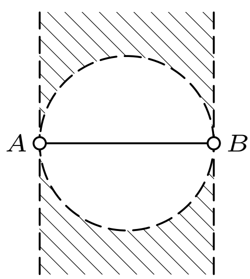
Remarks. 1. The points O, C, D lie on a straight line since $\angle AOC = \beta/2$ and $\angle AOD = \beta/2$ (prove it).

2. The statement is true for the inscribed circle as well (prove it).

Level B



Problem 1. We first draw the perpendicular to AB through the endpoint A. Clearly, $\angle A < 90°$ if and only if B and C lie on the same side of this line. Applying the same argument to B, we see that the locus of points C such that $\angle A < 90°$ and $\angle B < 90°$ is the strip bounded by the perpendiculars to AB at both A and B. (See figure on the right.)



Next we construct the circle with diameter AB. From Fact 14 we know that a point C is *outside* this circle if and only if it satisfies $\angle ACB < 90°$. Taking the intersection with the strip already found, we see that the locus of points C such that the ABC is acute is the shaded set to the left.

Finally we study the condition that the angle A is intermediate between the others, that is, either

$$\angle B \le \angle A \le \angle C \quad \text{or} \quad \angle C \le \angle A \le \angle B. \tag{1}$$

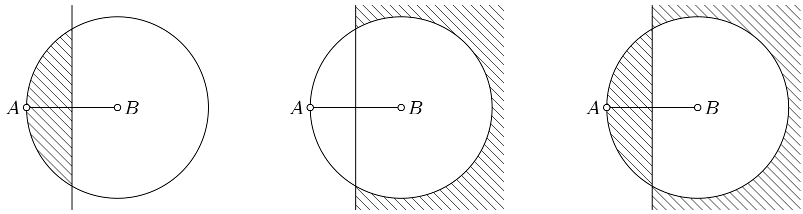
Since the greater angle of a triangle subtends the longer side, the condition $\angle B \le \angle A \le \angle C$ is equivalent to

$$AC \le BC \le AB.$$

The points equidistant from A and B are those on the perpendicular bisector L of the segment AB; therefore $AC \le BC$ if and only if C is in the half-plane determined by L and containing A.

At the same time, $BC \le AB$ if and only if C lies in the circle of center in B and radius AB. Thus, the locus of points satisfying $AC \le BC \le AB$

is the shaded set in diagram (a) below:



(a) (b) (c)

Similarly, the condition $\angle C \le \angle A \le \angle B$ is equivalent to $AB \le BC \le AC$, and the corresponding locus is depicted in diagram (b) of the previous page. The union of the sets in (a) and (b), shown in diagram (c), is therefore the locus of points C satisfying (1). There remains to draw the intersection of this locus and the one calculated immediately before (1).

Problem 2. We look at the first several terms of the sequence, in search of patterns. We find $x_3 = 9$, the least nonprime greater than $2x_2 - x_1 = 8$; then $x_4 = 14$ is the least nonprime greater than $2x_3 - x_2 = 12$ (because 13 is prime). Continuing we obtain $x_5 = 20$, $x_6 = 27$, and so on, where each time —after that exceptional 13—the least nonprime greater than $2x_{n-1} - x_{n-2}$ appears to be just $2x_{n-1} - x_{n-2} + 1$, a composite number already.

So we will *conjecture* that, for $n > 4$, the number $2x_{n-1} - x_{n-2} + 1$ is composite and so equals x_n. If this is so, we can write, for $n > 4$,

$$x_n - x_{n-1} = x_{n-1} - x_{n-2} + 1.$$

That is, the differences between successive elements of the series increase by one each time, forming an arithmetic progression: $x_4 = 14 = x_3 + 5$, $x_5 = 20 = x_4 + 6$, $x_6 = 27 = x_5 + 7$, ..., or, in compact form,

$$x_n = x_{n-1} + n + 1 \quad \text{for } n \ge 4.$$

If the differences form an arithmetic progression, the numbers x_n themselves can be calculated by summing up the progression. This would give

$$\begin{aligned} x_n &= x_3 + 5 + 6 + 7 + \cdots + (n+1) \\ &= \underbrace{2+3+4+5+6+7+\cdots+(n+1)}_{n \text{ terms}} = \tfrac{1}{2}n\bigl(2+(n+1)\bigr) = \tfrac{1}{2}n(n+3). \end{aligned}$$

(Here we broke x_3 down as $2 + 3 + 4$ to make the calculation less messy.)

Conjecturally, then, we conclude that

$$x_n = \tfrac{1}{2}n(n+3) \quad \text{for } n \ge 4. \tag{1}$$

There remains to prove that this is really the answer. We do it by complete induction.

Base of the induction. For $n = 4$, the equality (1) is true.

Induction step. Let (1) be true for $x_4, \dots, x_n$. We need to prove that $x_{n+1} = \frac{1}{2}(n+1)(n+4)$. We have

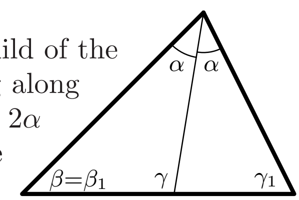
$$2x_n - x_{n-1} = 2 \cdot \tfrac{1}{2}n(n+3) - \tfrac{1}{2}(n-1)(n+2) = \tfrac{1}{2}(n+1)(n+4) - 1.$$

Thus, x_{n+1} will be equal to $\frac{1}{2}(n+1)(n+4)$ if this number is composite. This is indeed so: if n is even, $\frac{1}{2}(n+1)(n+4)$ has a factor $\frac{1}{2}(n+4) > 1$, and if n is odd, it has a factor $\frac{1}{2}(n+1) > 1$.

Now we just have to apply formula (1) to $n = 1000$, obtaining $x_{1000} = 501500$.

Problem 3. Assume that at some step we get a triangle similar to the initial one. All its angles are multiples of 20°.

Lemma. *All angles of the preceding triangle, and, more generally, of all preceding triangles are multiples of* 20°*.*



Proof. Let the triangle with angles α, β, γ be a child of the triangle with angles α_1, β_1, γ_1, obtained by cutting along an angle bisector. Arrange the labeling so that $\alpha_1 = 2\alpha$ and $\beta = \beta_1$, as in the figure. Since the sum of the angles of any triangle is 180°, we deduce that

$$\gamma_1 = 180^\circ - \alpha_1 - \beta_1 = (\alpha + \beta + \gamma) - 2\alpha - \beta = \gamma - \alpha.$$

Clearly, then, if α, β, γ are multiples of 20°, so are α_1, β_1, γ_1, the angles of the parent triangle (see Fact 5). But then the angles of the grandparent triangle are also multiples of 20°, and so on. □

This, however, cannot be true even after the first cut of the initial triangle: if we start with an angle of 20°, we get an angle of 10°, and if we start with the angle of 140° we get an angle of 70°. The contradiction shows that it is impossible to get a triangle similar to the initial one.

Remarks. 1. For any positive integer n, it is possible to find an initial triangle so the construction leads to a similar triangle after n appropriate bisector cuts, and no sooner.

2. For a similar situation, see Problem 01.B.4.

Problem 4. A classmate of Pete's can have between 0 and 28 friends. Of these 29 possibilities, we know that 28 occur. Thus, either there is a classmate who has 28 friends, or there is a classmate who has no friends. But if a classmate is friends with everyone, then everyone has at least one friend. So it's not possible for both 0 and 28 to occur: either the tally of friends is $1, \dots, 28$ for the various classmates, or it is $0, \dots, 27$.

Denote the classmate of Pete's with the most friends by A and the one with the least friends by B. In the first case just considered, A is everybody's friend, while B has only one friend, A. In the second case B has no friends, while A is friends with everyone except B. In either case, A is a friend of Pete's, and B is not.

Now let's send A and B to another class. Then Pete is left with 26 classmates and everybody has one fewer friend in this class than before. Thus each classmate still has a different number of friends in this class.

We again send the classmate with most friends and the one with least friends to another class. We can keep doing this until we have sent away 14 pairs of classmates. Each pair included exactly one friend of Pete's, so Pete had 14 friends in his class.

Remarks. 1. Several ideas work together toward the solution: friendship is assumed to be a symmetric relation; it's useful to look at extreme cases; and we are able to apply inductive descent.

2. There is one very short, but wrong solution: Let x be the number of Pete's friends. Now replace all friendships by nonfriendships, and all nonfriendships by friendships. Then Pete's classmates will again each have a different number of friends, so the conditions of the problem are still satisfied, meaning that Pete will again have x friends. But at the same time, we know that Pete now has $28 - x$ friends (his nonfriends in the original situation). Therefore

$$x = 28 - x.$$

Where is the mistake? (Hint: Have we shown that the problem has a unique solution?) Although wrong, this argument can point the way to the answer.

3. Solve the same problem if Pete has 27 classmates.

4. This is an extension of a problem you may have heard before: Prove that *in any group of more than one person, there are two people with the same number of friends within the group* (the number can be zero).

Problem 5. In the second identity set $y = z$. Then we get

$$(x * y) + y = x * (y * y) = x * 0.$$

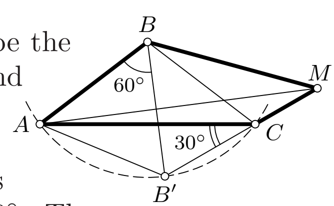
Thus, $x * y = x * 0 - y$. It remains to compute $x * 0$. For this, set $x = y = z$ in the second identity; we get

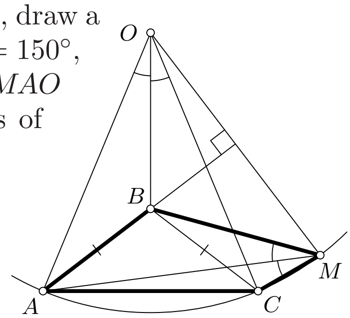
$$x * 0 = x * (x * x) = (x * x) + x = 0 + x = x.$$

Thus, $x * y = x * 0 - y = x - y$, so $1993 * 1935 = 1993 - 1935 = 58$.

Remark. Check that with $x * y = x - y$, both identities are indeed verified.

Problem 6. *First solution.* (See figure.) Let B' be the reflection of B in the line AM. Since $AB = AB'$ and $\angle BAB' = 2\angle BAM = 60°$, the triangle ABB' is equilateral. Hence, the points A, C, B' lie on a circle of center B. Since the inscribed angle is half the central angle, it follows that $\angle ACB' = 30°$. Then, since $\angle MCA = 150°$, the points C, B', M lie on the same line. Now, by construction, AM is the bisector of $\angle BMB'$, and so also of $\angle BMC$.





Second solution. Through the points A, C and M, draw a circle with center O (see figure). Since $\angle ACM = 150°$, the arc AM has measure $60°$, so the triangle MAO is equilateral. Point B lies on a symmetry axis of $\triangle MAO$, since $\angle BAM = 30°$. It follows that

$$\angle AMB = \angle AOB = \tfrac{1}{2}\angle AOC = \angle AMC,$$

where the second equality is a corollary of the congruence $\triangle ABO = \triangle CBO$.

Level C

Problem 1. The least period length of a decimal divides any other period length of that decimal; see Fact 4. (We regard any terminating decimal as having minimal period length 1).

Lemma. *If k is a (not necessarily minimal) period length for each of two decimals P and Q, then k is also a period length for $P+Q$ and $P-Q$.*

Proof. Recall (Fact 13) that a recurring decimal P with period k can be written in the form

$$P = \frac{X}{10^l(10^k-1)},$$

where X is an integer. Similarly, we can write

$$Q = \frac{Y}{10^m(10^k-1)},$$

where Y is an integer. Without loss of generality we may assume that $l \geq m$. We obtain

$$P \pm Q = \frac{X \pm 10^{l-m}Y}{10^l(10^k-1)},$$

where again the numerator is an integer, so the decimals corresponding to $P+Q$ and $P-Q$ are recur with periods of length k. □

Now we can solve the problem. We know that A has least period 6 and B has least period 12. The lemma implies that 12 is a period length of $A+B$, so the divisors of 12 are the candidates for the least period of $A+B$. But 6 cannot be a period length of $A+B$, otherwise $B=(A+B)-A$ would have a period of length 6, contradicting the assumption. Hence the least period of $A+B$ cannot be a divisor of 6.

Two possibilities remain for the least period of $A+B$: 12 and 4. Both options are possible:

$A = 0.(000001)$, $B = 0.(000000000001)$, $A+B = 0.(000001000002)$;

$A = 0.(000001)$, $B = 0.(011100110110)$, $A+B = 0.(0111)$.

Remarks. 1. How would someone come up with this last example? By working backwards! Take *any* decimal of least period 4 and call it $A+B$. Now subtract any decimal B of least period 6. The result has least period 12. (Why?)

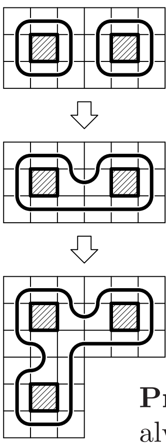
2. We can find all possible least periods of a sum of two decimals. Let m, n, k be the least periods of the decimals P, Q and $P+Q$, respectively. Then k divides the least common multiple $\operatorname{lcm}(m,n)$ of m and n, by the lemma; but at the same time m divides $\operatorname{lcm}(n,k)$, and n divides $\operatorname{lcm}(m,k)$. Let $p_1,\dots,p_s$ be all the prime divisors of $\operatorname{lcm}(m,n)$, and write

$$m = p_1^{\alpha_1}\dots p_s^{\alpha_s}, \quad n = p_1^{\beta_1}\dots p_s^{\beta_s},$$

where the exponent are allowed to be 0; see Fact 10. The preceding arguments imply that $k = p_1^{\gamma_1}\dots p_s^{\gamma_s}$, where

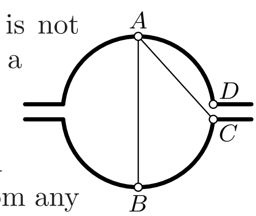
$$\gamma_i = \begin{cases} \max(\alpha_i,\beta_i) & \text{if } \alpha_i \neq \beta_i, \\ \text{any number from 0 through } \alpha_i & \text{if } \alpha_i = \beta_i. \end{cases}$$

It can be shown that the opposite is also true: any such k can be a least period of a sum of two decimals whose least periods are m and n, respectively.

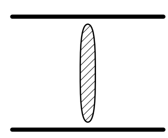


Problem 2. We can certainly tour the eight rooms in any hall, going clockwise, say. We can also combine tours of two adjacent halls (top diagram) by using two of the three doors that separate the two halls (middle diagram).

Now the key observation is that we can always add a hall to our tour, so long as it is shares a wall with a hall already on the tour. For example, the bottom diagram adds a third hall to the first two. We can continue adding one hall at a time, until the tour includes all the halls of the castle.



Problem 3. The answer to (a) is that it is not always possible; see figure on the right for a counterexample. In it, $AC = 1000\,\mathrm{m}$, $AB > 1400\,\mathrm{m}$, $CD = 1\,\mathrm{m}$. The segment AB divides the river into two parts, and boat going down the river end to end must cross this segment at some point. The distance from any point of AB to one of the banks exceeds $700\,\mathrm{m}$.



Part (b) turned out to be unexpectedly difficult, unless islands are allowed, in which case a counterexample is not hard to find (see figure on the right; the river has width just under 1 km).

The intention of the authors of the problem was to not allow islands, and we, the authors of the book, were at first unable to solve it under this constraint. A contest was announced, and a solution was found through the joint efforts of A. Akopyan, V. Kleptsyn, M. Prokhorova, and the authors. It turned out to be more of a research-level mathematics problem than an Olympiad problem!

Before presenting a solution in the next 6 pages, we list its key ideas:

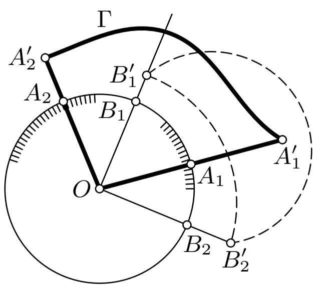
- Show that no disc of radius 750 m and center in the river lies entirely on water.
- Deduce that if the swimming distance from a point in the river to one bank is at least 750 m, the point lies within 750 m of the other bank.
- Use the (possibly very complicated) boundary of the set of points that lie within 750 m of the left bank to prove that there is a path that remains within 800 m of both banks. (The previous statement shows that every point on the boundary is within 750 m from each bank.)

Detailed solution. Consider an arbitrary disk lying entirely on the water and points A_1, A_2, B_1, and B_2 on its boundary. Suppose that the points B_1 and B_2 separate A_1 and A_2: that is, moving around the circle, we cannot get from A_1 to A_2 without encountering B_1 or B_2.

Denote by O the center of the circle and by A_1' the first intersection of the ray OA_1 with the bank. Points A_2', B_1', and B_2' are defined similarly. Consider the *water boundary*, which is the union of the river's left bank, the right bank, and the edges of the lakes. This is a simple closed curve made of line segments and arcs of circles. (A *simple curve* is one that does not have self-intersections.) We will use repeatedly the topological statement below:

Lemma 0. *The points B_1' and B_2' separate the points A_1' and A_2', i.e., it is impossible to get from A_1' to A_2' moving along the water boundary and not encountering B_1' or B_2'.*

Proof. Suppose that the statement of the lemma is false. Denote by Γ a portion of the water boundary joining A_1' and A_2' but avoiding B_1' and B_2' (see figure). Clearly, the curve Γ must cross one of the rays OB_1' or OB_2' — let it be OB_1' — at a point further from the center than B_1'. Consider the contour consisting of Γ and the segments OA_1' and OA_2'. This contour separates the points B_1' and B_2', and therefore, the water boundary between B_1' and B_2' must cross it. But this is impossible, because A_1' and A_2' were the *first* points of the bank on their respective rays. The proof is complete. □



Remark. We have used the famous *Jordan curve theorem*: *every closed plane curve without self-intersections divides the plane into two parts*, an inside and an outside. For curves consisting of segments and circle arcs, its proof is not very difficult. For the general case, see for example [615, pp. 100–109].

Lemma 1. *Under the assumptions of the problem, there is no disk of radius* 750 m *with center in the river that lies entirely on water.*

Proof. Suppose that such a disk exists and denote its center by O. We prove there is a point on one of the banks such that the distance from this point to the other bank exceeds 1000 m.

Let's color the water boundary as follows: blue for the right bank of the river, green for the left bank, yellow for the edge of one lake, and white for the edge of the other. We agree that lakes take priority over river banks where the two meet. Blue and green points will be called *dark*, while yellow and white points are *light*.

Each point of the circle also gets a color, according to where the ray from O to that point first hits the water boundary. This divides the circle into colored arcs — possibly only one, equal to the whole circle. (Each arc can be open, half-open, or closed; a single point, which might *a priori* occur, is considered a closed arc.)

Step 1. *There is at most one arc of each color.* Indeed, suppose you can take two points A_1 and A_2 on *different* green arcs, say. These points split the circle into two pieces, on each of which we take a point of a color other than green. Denote these points by B_1 and B_2. Now Lemma 0 says that the points on the water boundary corresponding to B_1 and B_2 separate those corresponding to A_1 and A_2; but this is impossible since each color on the water boundary occupies a connected set.

Remark. An experienced reader may argue that the sets of points of each color are not necessarily finite unions of arcs. However it follows from the reasoning above that these sets are connected, and a connected set on a circle is an arc.

Step 2. *If a blue and a green arc are present, the white and yellow arcs cannot be contiguous.*

Proof. Suppose the arcs are arranged, for instance, in the order white, yellow, green, and blue. To obtain a contradiction, we take four points A_1, A_2, B_1, and B_2 colored white, green, yellow, and blue, respectively, and apply Lemma 0 to them. Other possibilities are treated analogously. □

Step 3. *Each light-colored arc is less than* $180°$, because the center of the circle is outside the lake by assumption, and so it can be separated from the lake by a line. It follows that there is at least one dark arc.

Step 4. *Either each of the dark arcs measures* $180°$ (this case will be called *exceptional*) *or there are points A_1, A_2, and B on the circle with the following properties*:

(1) *A_1 and A_2 are diametrically opposite points.*
(2) *B is the midpoint of one of the arcs A_1A_2.*
(3) *B is dark.*
(4) *A_1 and A_2 are either light or the same color as B.* (This case will be called *general*.)

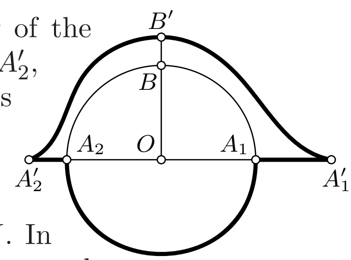
Proof. We can assume that both blue and green arcs do exist and each of them measures less than $180°$; all the other cases are trivial. First, let us suppose that both light arcs exist. Using Step 2, we see that there is a diameter d_1 joining white and yellow points. Consider the perpendicular diameter d_2. If one of the endpoints of this diameter is dark, then we can take it as the point B, and the endpoints of d_1 as the points A_1 and A_2.

If both endpoints of d_2 are light, the diameters d_1 and d_2 divide the circle into four parts, each of the dark arcs lying entirely in one part. The parts that contain dark arcs are opposite, because otherwise one of the light arcs would be at least 180°. Now we can take any dark point as B.

The case of one light and two dark arcs is left to the reader. □

Step 5. Consider the general case first. We'll assume that the point B is blue. Define the points A_1', A_2', and B' on the water boundary as before. Denote by Γ the portion of the water boundary between A_1' and A_2' that contains B'. By choosing if necessary new points A_1, A_2 and B satisfying conditions (1)–(4) above, *we may assume that* Γ *contains no point from the left bank.*

Proof. The points A_1' and A_2' do not lie on the left bank; therefore, either there are no points of the left bank on Γ or Γ contains the entire left bank. In the first case the statement is true; in the second case Γ contains the entire edge of one of the lakes. Assume this lake is yellow. There are two possibilities: either both points A_1 and A_2 are blue or one of them is blue and the other is white. If A_1 and A_2 are blue, then, by Lemma 0, the entire circle is blue. By using Lemma 0 again, it is not difficult to check that it will suffice to replace the point B by the diametrically opposite one. If B is white, then the circle consists only of the yellow and blue arcs, so the blue arc measures more than 180°. But then we can choose new points A_i and B, all of them blue, and reduce the consideration to the previous case, completing the proof. □



Now construct the closed curve consisting of the water boundary Γ, the segments A_1A_1' and A_2A_2', and the semicircle joining A_1 and A_2 that does not contain B (see figure).

Since Γ is disjoint from the left bank, the shortest path from B' to the left bank crosses the portion $A_1'A_1A_2A_2'$ of Γ at a certain point X. In any case, we have $\angle B'OX \geq 90°$, $B'O \geq 750\,\text{m}$, and $XO \geq 750\,\text{m}$. Hence, by the law of cosines, we have

$$B'X \geq \sqrt{B'O^2 + XO^2} \geq 750\sqrt{2} > 1000\,\text{m},$$

contrary to the assumption.

It remains to consider the exceptional case. In this case, we cannot choose two blue diametrically opposite points, but we can take them to be almost so. More exactly, we'll choose the points A_1 and A_2 so as to ensure the inequality $\cos\alpha < 1/9$, where 2α is the smaller of the arcs A_1A_2. Then the previous inequalities will be replaced by

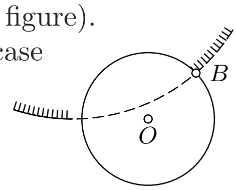
$$B'X \geq \sqrt{B'O^2 + XO^2 - \frac{2B'O \cdot XO}{9}} \geq \sqrt{B'O^2 + XO^2 - \frac{B'O^2 + XO^2}{9}}$$
$$\geq \sqrt{\tfrac{8}{9}}\sqrt{750^2 + 750^2} = 1000 \text{ m}.$$

This completes the proof of Lemma 1. □

Lemma 2. *Suppose the (swimming) distance from a point O on the river to the left bank is at least* 750 *m. Then the distance to the right bank is at most* 750 *m.*

Proof. Let B be a point of the water boundary closest to O. (If there is more than one such point, any of them can be chosen.) The open segment OB contains no land, so the swimming distance from O to B is just the length of OB. By Lemma 1, this is less than 750 m, which means, by our lemma's assumption, that B cannot be on the left bank. If B is on the right bank, the statement is proved. It remains to consider the case in which B is a point on a lake bank. This lake is a disk; denote it by K_1. Let the lakes meet the left bank at points P and Q, and the right bank at R and S.

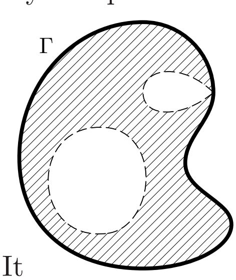
Let K_2 be the disk of radius OB with center O. There are no points of the bank inside this disk. Therefore, either the disks K_1 and K_2 touch each other or B is one of the points P, Q, R, or S (see figure). In the second case, B is a point of the bank, and this case has already been considered. In the first case, it is not difficult to see that B is one of the points P, Q, R, and S as well (otherwise we would reach the lake bank from outside). This completes the proof of the lemma. □



All this was preparation. We now give an outline of the solution of the problem proper.

Let X be the set of points lying on water and whose swimming distance to the left bank is less than 800 m. This set is connected: it is possible to swim from any point in X to the left bank along a path that remains within X, and therefore it's possible to swim between any two points of X while remaining in X.

Let Γ be the *exterior boundary of* X, defined as the component of the boundary consisting of the points from which we can move "to infinity" — that is, arbitrarily far away — along a path outside X. (For example, the exterior boundary of the gray set on the right is the thick curve.)



Then Γ *is a closed simple curve* (see Remark 1). It consists of points of two types: points whose distance to the left bank *equals* 800 m, and points on the water boundary.

The entire left bank, which we denote by Γ_1, is contained in Γ. Indeed, let A be a point of the left bank; if we move a little distance from it into the river, we obtain a point inside X, and if do so into the land, we obtain a point outside X. Therefore, A belongs to the boundary of X. And it belongs to the exterior boundary, because we can walk from A to infinity by land.

Thus, Γ is representable as the union of the curve Γ_1 and its complement $\Gamma\setminus\Gamma_1$. Notice that $\Gamma\setminus\Gamma_1$ joins the endpoints of the left bank, i.e., it joins the lakes. Let us prove that the pilot can steer the boat along this complement. It will suffice to prove that the distance from any point $\Gamma\setminus\Gamma_1$ to each of the banks is at most 800 m, if only this point does not belong to the lakes.

Consider an arbitrary point of $\Gamma\setminus\Gamma_1$. Its distance to the left bank is at most 800 m. If it is a point of the right bank, the distance to the right bank is less than 800 m. If it is a point in the river, then the statement follows from Lemma 2. This solves the problem.

Remarks. 1. Why was this only an outline? Because we have not proved that Γ is a simple closed curve. A priori, the boundary of a set X — even one assumed to be open — can be very complicated. (In the example known as the "lakes of Wada", given by K. Yoneyama in 1917, three connected open sets have each the *same* boundary, and their boundary equals the intersection of the complements of the three! See [615, p. 100–101].) The boundary is only guaranteed to be a simple closed curve if X satisfies certain properties. Our X does satisfy them, but this is not so easy to prove. In any case, Γ is not necessarily the union of straight line segments and arcs, as one might naively assume! (See Remark 2 on the next page.)

A simpler approach requires approximating the banks — which, by assumption, do consist of segments and arcs — by polygonal lines. A detailed presentation of this argument would take many pages, so we split it into a series of relatively simple assertions (called Problems below) to be proved by the reader.

Denote by $\varepsilon_0 > 0$ the minimum swimming distance between the points A and B, where A is on the left bank of the river and B is on the right bank. (If you understand compactness and continuity, think about why ε_0 exists and is positive.) Let $\varepsilon = \min(\varepsilon_0/3, 50\text{ m})$.

Problem. *It is possible to sail from P to Q along a polygonal line L_1 so that the swimming distance to the left bank never exceeds ε.*

Let us fill up the area of water between the path L_1 and the left bank with sand. In the same way, we draw a polygonal path L_2 along the right bank and fill up with sand the corresponding area of water. Let X' be the set of points on the water whose swimming distance from the polygonal path L_1 is less than $d = 750$ meters.

We'll say that an open bounded domain X is *nice* if its boundary is the union of finitely many segments and arcs of circle.

Problem. *The exterior boundary of a nice and connected open set can be represented as $\bigcup_{i=1}^{M}\Gamma_i$, where*

- *each Γ_i is either a segment or an arc, and*
- *there are pairwise distinct points F_i, $i = 1, \ldots, M$, such that*
  - *Γ_i joins F_i to F_{i+1},*
  - *Γ_M joins F_M to F_1, and*
  - *each Γ_i is disjoint from the interior of Γ_j for $i \neq j$.*

(*Hint.* Consider the boundary of the domain as a graph on the plane whose edges are segments and arcs. Then the exterior boundary is a subgraph of this graph — for this notion, see Fact 3 on page 194. Consider a cycle in this subgraph.)

Problem. *Deduce the statement of the problem from the fact that X' is a nice domain.*

It remains to prove that X' is a nice domain.

Define a *basic path* to be any polygonal line $A_1A_2 \dots A_kC$ such that

(1) the A_i are nodes of L_2,
(2) all edges of the path lie on water, and
(3) C is either a vertex of L_1 or the base of the perpendicular dropped from A_k onto one of the edges of L_1.

Clearly, the number of basic paths is finite.

Problem. *The shortest swimming path from any point B to the left bank is either a straight line segment or the union of a segment and a basic path.* (*Hint.* Fix a point B and consider a shortest path of the specified form. Assume that there exists a shorter swimming path; apply descending induction on the number of vertices visited by this path.)

Denote by $\gamma_1, \dots, \gamma_N$ all basic paths, by $d_1, \dots, d_N$ their lengths, and by $E_1, \dots, E_N$ their initial points.

Problem. *The shortest swimming distance from a point B to the left bank is either the distance along a line or the smallest of the quantities*

$$BE_i + d_i,$$

where the minimum is taken over the i for which BE_i lies entirely on water.

Let X_i be the set of points for which the segment BE_i lies on water and $BE_i < d - d_i$. Let Y_i the set of points from which it is possible to sail along a line to the i-th vertex of the path L_1 having covered a distance of at most d. Finally, let Z_i be the set of points from which it is possible to drop a perpendicular onto the i-th edge of the path L_1 lying entirely on water, its length being at most d.

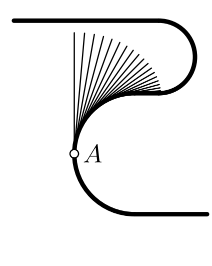
Problem. *X' is the union of the sets X_i, Y_i, and Z_i.*

Problem. *Each of the sets X_i and Y_i is the intersection of a polygon and a disk and each of the sets Z_i is a polygon.*

Problem. *X' is a nice domain.* (*Hint.* Prove that the union of two nice domains is a nice domain.)

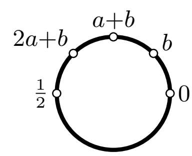
And now... Congratulations!

2. The set Γ is not necessarily the union of segments and arcs. For instance, suppose that the right bank is a segment and the left bank contains a circle of radius 200 m. Let A be a point of the circle such that the tangent at this point is perpendicular to the right bank. Suppose that the distance from A to the right bank is 400 m. Draw a tangent from each point X of the upper semicircle and mark off the distance of 400 m without the length of the arc AX. The curve thus obtained is called the *involute* of a circle. The distance from the points of the involute shown in the figure to the right bank is equal to 800 m, and so a part of this involute is contained in Γ.



3. A. Akopyan has proposed the following elegant argument for the overall proof: Let us color white the points on the water at most 800 m away from both banks and black, all the other points. If we can sail from one lake to the other along a white path, we're done; otherwise the black points form a *bridge* between the right and left banks. One can show that there is a point in the bridge at least 800 m away from each of the banks, and then apply Lemma 2. Unfortunately, the statement about the bridge is not elementary either.
4. The idea of another solution, based on the consideration of points at which the banks subtend equal angles, was proposed by D. Piontkovsky.

Problem 4. Since we're taking fractional parts of numbers, it's a good idea to visualize the real line "rolled up" into a circle of unit length, so numbers with the same fractional part correspond to the same point on the circle. (See figure on the right, and compare the remark after the solution of Problem 97.C.6.) It is clear that $p_n = 0$ if $x_n = \{an + b\}$ lies on the upper semicircle $[0, \frac{1}{2})$, and $p_n = 1$ if x_n lies on the lower semicircle. Furthermore, $x_n = \{an + b\}$ is the point on the circle obtained from $\{b\}$ by n consecutive rotations through the arc $\{a\}$.



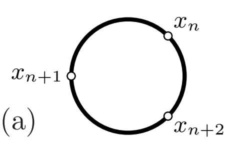
(a) A rational value of a leads to a periodic pattern for the sequence p_n. (Why?) Trying out $b = 0$ and the simplest possible fractions for a, we see that all 4-tuples of 0s and 1s can occur:

- The "bigon" with $a = \frac{1}{2}$ gives the sequence 010101..., so 0101 can occur. (We only list 4-tuples starting with 0, since we know the complementary ones can be obtained by choosing $b = \frac{1}{2}$ instead of $b = 0$, leading to the replacement of p_n by $1 - p_n$.)
- The equilateral triangle with $a = \frac{1}{3}$ gives the sequence 001001..., so we get 0010 and 0100.
- The square with $a = \frac{1}{4}$ gives 0011 and 0110.
- The octagon with $a = \frac{1}{8}$ gives 0000, 0001, 0011 again, and 0111.

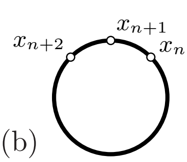
(b) *First solution.* We will prove that the string 00010 cannot be realized for any a and b. (In the second solution below we give a general condition necessary for a string to be realizable.)

Consider three consecutive terms of the sequence: x_n, x_{n+1} and x_{n+2}. If the points x_n and x_{n+2} are diametrically opposed, each point is obtained from the previous one by a 90° rotation, and we're in the situation of the square in part (a); the sequence p_n cannot contain three zeros in a row.

If the points x_n and x_{n+2} are not diametrically opposed, they divide the circle into two distinct arcs, and either x_{n+1} lies on the longer arc, as in diagram (a), or x_{n+1} lies on the shorter arc, as in (b).



(a)



(b)

Suppose x_{n+1} lies on the longer arc. Then any other three consecutive points x_m, x_{m+1} and x_{m+2} are similarly situated, being obtained from the points x_n, x_{n+1} and x_{n+2} by the same rotation. This means that three such points cannot appear in the upper half-circle; that is, the substring 000 cannot appear in the sequence p_n.

Now suppose instead that x_{n+1} lies on the shorter arc $x_n x_{n+2}$. Then any other three consecutive points x_m, x_{m+1} and x_{m+2} are similarly situated. In particular, if x_m and x_{m+2} belong to the upper half-circle, so does x_{m+1}; that is, the substring 010 cannot appear in the sequence p_n.

We have considered all the scenarios and proved that the string 00010 cannot be encountered.

Second solution. Let's look at runs of consecutive 0s and 1s in the sequence $p_0, p_1, p_2, \ldots$. We claim that *all such runs have either the same length or lengths differing by* 1, except that the first run may be short (it can start in the middle, so to speak). In particular, the string 00010 can never occur for any a and b, because it would mean the sequence has a run of 0s of length at least 3, but also a run of 1s of length 1.

The reason the runs have almost uniform length is that the spacing between x_n and x_{n+1} around the circle is the same for all n. Indeed, suppose first that $0 < a \le \frac{1}{2}$ and let $i \ge 1$ be the integer uniquely defined by

$$i \le \frac{1}{2a} < i{+}1, \quad \text{or equivalently,} \quad i\,a \le \tfrac{1}{2} < (i{+}1)\,a. \tag{1}$$

We now show that a change from 0 to 1 must be followed by a change from 1 to 0 after exactly i or $i{+}1$ entries. More formally, suppose $p_n = 0$ and $p_{n+1} = 1$; that is, x_n belongs to the upper semicircle $[0, \frac{1}{2})$, while $x_{n+1} = x_n + a$ belongs to the lower semicircle $[\frac{1}{2}, 1)$. The third inequality in (1) gives

$$a < 2a < \cdots < i\,a \le \tfrac{1}{2}.$$

It follows that x_{n+1}, x_{n+2}, …, x_{n+i} all lie in $[\frac{1}{2}, 1)$, giving i consecutive 1s. If we had $p_{n+i+1} = p_{n+i+2} = 1$, there would be $i + 1$ consecutive intervals of length a inside the lower semicircle: from x_n to x_{n+1}, then to x_{n+2}, and so on up to x_{n+i+2}. But this is impossible by the last inequality in (1). Therefore the run of 1s stops at length i or $i + 1$.

A completely analogous argument shows that a change from l to 0 must be followed by a change from 0 to 1 after exactly i or $i{+}1$ entries. This proves our claim for $a \in (0, \frac{1}{2}]$. If $a \in (\frac{1}{2}, 1)$ we replace a by $1 - a$, which corresponds to a rotation through the same angle but in the opposite direction. We needn't consider values of a outside $[0, 1)$ because only the fractional part of a matters. Finally, for $a = 0$, there is only one run, of infinite length. So our claim is proved in all cases.

Remarks. 1. This problem is relevant in symbolic dynamics; see [701].

2. The original Olympiad problem said "… the sequence determined by some a and b", which is perhaps ambiguous: the question might be whether all possible quadruples of 0s and 1s can occur for some *fixed* choice of a and b. The second

solution to part (b) shows that the answer is no: any values of a and b that allow the string 0000 cannot allow the string 0100.

Problem 5. Let m be the number of plants in a certain good classifier. Let us estimate the total number S of distinctions between all pairs of plants with respect to all features. There are $\frac{1}{2}m(m-1)$ pairs of plants, and each pair differs in at least 51 features, so

$$S \geq 51 \cdot \tfrac{1}{2}m(m-1).$$

There is another way to look at S. Let m_i be the number of plants having feature i. The number of pairs of plants that can be distinguished by means of feature i is $(m-m_i)m_i$. Summing over all the features, we obtain the total number S of distinctions:

$$S = \sum_{i=1}^{100}(m-m_i)m_i.$$

Now, the arithmetic mean of $m-m_i$ and m_i is $\frac{1}{2}m$, so the inequality between the arithmetic and geometric means (Fact 26) gives $(m - m_i)m_i \leq \frac{1}{4}m^2$. Therefore $S \leq 100 \cdot \frac{1}{4}m^2 = 25m^2$. Combining this with the earlier bound, we obtain

$$51 \cdot \tfrac{1}{2}m(m-1) \leq S \leq 25m^2. \qquad (1)$$

Subtracting $25m^2$ from the first and last expressions and simplifying we get $\frac{1}{2}m(m-51) \leq 0$. Hence $m \leq 51$.

It remains to prove that $m \neq 51$. If $m = 51$, we have a strict inequality

$$m_i(m - m_i) < \tfrac{1}{4}m^2$$

(since we have an integer on the left and a fraction on the right). That means the second inequality in (1) is strict, implying $m < 51$. This contradiction implies that a good classifier cannot describe more than 50 plants.

Remarks. 1. One might be tempted to guess that a good classifier can describe 50 plants, but with a bit more work one can show that this is far from true. What we do is extend the classifier with one extra feature, which we declare present if and only if an even number of the original 100 features were present. This new classifier has 101 features and any pair differs by at least 52 of them: if a pair differs in 51 of the original features, it must also differ in the new feature. (Why?)

Now the same arguments as above yield

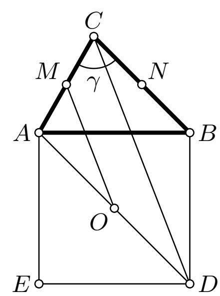
$$52 \cdot \tfrac{1}{2}m(m-1) \leq S \leq 101 \cdot \tfrac{1}{4}m^2,$$

which leads to $m \leq 34$. Thus, the new classifier (and hence the initial one) can describe at most 34 plants.

2. This problem is related to *error correcting codes*. Replace plants by messages and descriptions of features by length-n strings of bits (0s and 1s). The classifier — which is now a collection of m strings — is said to be a *code* of length n. The minimum number d of differences between two sequences in the code is the *code distance*; in our problem $d = 51$.

If we take a message in the code and distort it arbitrarily by flipping no more than $\frac{1}{2}(d-1)$ positions, we are still able to recover the original message, simply by selecting the message that shares the most bits with the distorted one. (There will be at most $\frac{1}{2}(d-1)$ differences, whereas the comparison with any other string will give at least $d - \frac{1}{2}(d-1) > \frac{1}{2}(d-1)$ differences.) This is why the code is said to be *error-correcting*.

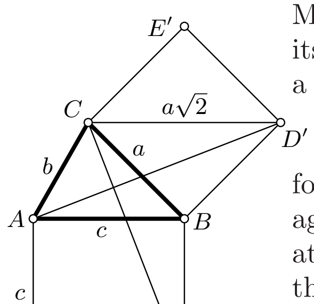
One important open problem of code theory is finding the maximal size (number of different messages) of a length-n error-correcting code with code distance d, for arbitrary n and d. A famous result in this direction is the Plotkin–Levenshtein theorem, which establishes an upper bound (called the *Plotkin bound*) in the case $d > \frac{1}{2}n$, and provides certain natural conditions that guarantee the bound can be achieved. In our problem we have $n = 100$ and $d = 51$, so Plotkin's bound applies, and its value is 34. This bound is achievable: there does exist a code with 34 messages.



Problem 6. Let the remaining vertices of the square on side AB be D and E, so the square is $ABDE$, and set $\gamma = \angle ACB$. (See figure on the right.) Applying the intercept theorem to the triangle ADC we see that $CD = 2\,OM$; similarly, $CE = 2\,ON$. Therefore it suffices to find the maximum of $CD + CE = 2(OM + ON)$.

First solution. On side BC of $\triangle ABC$, construct a square $CBD'E'$ outwards. (See figure below.) Triangles ABD' and DBC have two sides and the included angle equal, so $CD = AD'$.

In the triangle ACD', two sides are known: $AC = b$ and $CD' = a\sqrt{2}$. Moreover, $\angle ACD' = \gamma + 45°$. The side AD' attains its maximal value when the triangle degenerates into a segment, so



$$\max(CD) = \max(AD') = b + a\sqrt{2}$$

for $\gamma = 135°$. Similarly, we have $\max(CE) = a + b\sqrt{2}$, again for $\gamma = 135°$. Thus, each of OM and ON attains its maximum when $\gamma = 135°$, and so does their sum:

$$\max(OM + ON) = \frac{1+\sqrt{2}}{2}(a+b).$$

Second solution. Let $\angle CAB = \alpha$, $\angle ABC = \beta$, $c = AB$, $d = CD$, $e = CE$. The law of cosines for $\triangle ABC$ gives

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

Next we apply the law of cosines to $\triangle AEC$:

$$e^2 = b^2 + c^2 - 2bc\cos(90° + \alpha) = b^2 + c^2 + 2bc\sin\alpha,$$

since $\cos(90° + \alpha) = -\sin\alpha$. Substituting c^2 from the first formula into the second, we get

$$e^2 = 2b^2 + a^2 - 2ab\cos\gamma + 2bc\sin\alpha.$$

The law of sines for $\triangle ABC$ implies that $\sin\alpha = (a/c)\sin\gamma$. Therefore

$$e^2 = 2b^2 + a^2 + 2ab\,(\sin\gamma - \cos\gamma).$$

Similarly,

$$d^2 = 2a^2 + b^2 + 2ab\,(\sin\gamma - \cos\gamma).$$

Hence both e and d attain their maximum values when $\sin\gamma - \cos\gamma$ does, which is to say, when $\gamma = 135°$. Hence the maximum of $e+d$ is also attained for $\gamma = 135°$, and it equals $\frac{1}{2}(1+\sqrt{2})(a+b)$.

Level D

Problem 1. If $\tan(\alpha+\beta)$ is defined, then

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\cdot\tan\beta} = \frac{p}{1 - \tan\alpha\cdot\tan\beta}. \qquad (1)$$

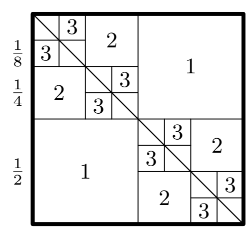
The product of tangents is related with p and q as follows:

$$q = \frac{1}{\tan\alpha} + \frac{1}{\tan\beta} = \frac{\tan\alpha + \tan\beta}{\tan\alpha\cdot\tan\beta} = \frac{p}{\tan\alpha\cdot\tan\beta}. \qquad (2)$$

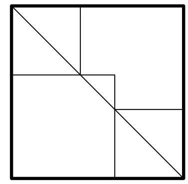
We deduce from (2) that either p and q are both zero or they are both nonzero; therefore we have only these cases to consider:

1. If $p = 0 = q$, then (1) implies $\tan(\alpha+\beta) = 0$. We have to verify here that the denominator of (1) does not vanish. Indeed, since $p = 0$, we have
$$\tan\alpha = -\tan\beta, \quad \text{so} \quad 1 - \tan\alpha\cdot\tan\beta = 1 + \tan^2\alpha > 0.$$
2. If $p \neq 0$, $q \neq 0$ and $p \neq q$, then (2) implies $\tan\alpha\cdot\tan\beta = p/q$, so (1) implies
$$\tan(\alpha+\beta) = \frac{pq}{q-p}.$$
3. If $p \neq 0$, $q \neq 0$ but $p = q$, then $\tan(\alpha+\beta)$ is not defined.

Problem 2. We will construct a subdivision into squares satisfying the condition of the problem. We first divide the unit square into four equal squares. The little squares that intersect the main diagonal only at a vertex will be called level-1 squares. We subdivide each of the remaining squares into four equal squares of side $\frac{1}{4}$. The little squares of side $\frac{1}{4}$ that intersect the main diagonal only at a vertex will be called level-2 squares. We continue in this way (see figure) until we have 500 levels of squares.



There are 2^k squares at level k, each with side 2^{-k}. Hence the total perimeter of all level-k squares is 4, and the total perimeter of all squares intersecting the diagonal is $4\cdot 500 > 1993$.



Remarks. 1. A stronger result is in fact true: *The unit square can be partitioned into squares in such a way that the sum of perimeters of squares intersecting the main diagonal in a segment exceeds any given number.* The construction is a modification of the previous one; we make the under-the-diagonal level-1 square have side $\frac{3}{5}$, say, instead of $\frac{1}{2}$, while the complement is subdivided into two squares of side $\frac{2}{5}$ along the diagonal, plus an irregular area that we leave aside, knowing that it can be subdivided into tiny squares (since $\frac{3}{5}$ is a rational number).

We then repeat the procedure on the two new squares along the diagonal to get level-2 squares, and so on. All level-k squares now have side $(\frac{2}{5})^k$, instead of $(\frac{1}{2})^k$ as before, and there are 2^{k-1} of them, instead of 2^k. So now the total perimeter of level-k squares is $2 \cdot (\frac{4}{5})^k$; that is, it decreases in a geometric progression instead of being the same for all levels. But the ratio of the progression can be made arbitrarily close to 1, by replacing the number $\frac{3}{5}$ by some rational number very close to $\frac{1}{2}$. So the sum of perimeters can be made as large as desired.

2. This problem arose during a lecture of the illustrious mathematician N. N. Luzin, when he wanted to shorten the proof of a theorem of Cauchy (Luzin loved to improvise). Luzin conjectured: *Fix a curve of a bounded length in the unit square and consider a partition of the square into little squares. The total perimeter of the little squares that intersect the curve is bounded by a constant depending on the curve only.* A. N. Kolmogorov, who was to become just as famous a mathematician, was at the lecture and soon constructed a counterexample.

Problem 3. For $n = 2$ the answer is obviously 1. So assume $n \geq 3$.

The desired number is at most n, because we can exhibit an arrangement of points generating only n pairwise nonparallel lines: the vertices of a regular n-gon,

We prove this by showing that there are as many nonparallel lines as there are axes of symmetry of the polygon. To each side and each diagonal, we assign an axis of symmetry: the perpendicular bisector of this side or diagonal. Two sides or diagonals have the same perpendicular bisector if and only if they are parallel. Therefore we just need to count the axes of symmetry of a regular n-gon.

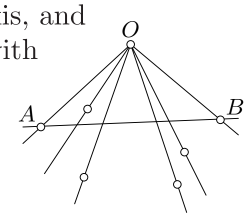
For n odd, each axis of symmetry passes through a unique vertex. Hence, the total number of axes of symmetry is n.

For n even, an axis of symmetry passes either through a pair of opposite vertices or through the midpoints of opposite sides. There are $n/2$ axes of either type. Hence, the total number of axes of symmetry is again n.

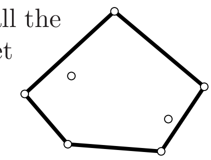
Now we prove the converse: *The desired number is at least* n. That is, for any arrangement of n points, no three of which lie on a line, we can always find n pairwise nonparallel lines.

It is easy to find $n - 1$ lines: just take a point and consider all the lines from it to the other points. It is a bit harder to construct the n-th line.

One method is to introduce cartesian coordinates on the plane. Among the n points, take one, say O, having highest y-coordinate, and move the origin there. Among the other $n-1$ points, choose A such that the ray OA makes the biggest possible angle with the positive x-axis, and B such that the ray OB makes the least possible angle with the positive x-axis. All the rays connecting O with the other points lie inside the angle AOB, by our choice of A and B (see figure). Thus they must intersect the segment AB and hence cannot be parallel to it. Now we just take AB as the n-th line.



Remarks. 1. The points A, O and B are adjacent vertices of the *convex hull* of the n points, that is to say, the smallest convex set containing all the points. (It can be shown that the convex hull of a finite set is a polygon whose vertices are contained in the set. The convex hull of the seven points in the preceding figure has five vertices; see figure on the right.)



2. The given noncollinearity condition cannot be replaced by the weaker one that the points do not all lie on a single line. For example, for the set consisting of the vertices of a regular $2k$-gon and its center, there are only $2k$ nonparallel lines.

Problem 4. Clearly, the worst-case scenario is when all marble populations occur at the start; that is, we have n boxes—where $n = 460$ in part (a) and $n = 461$ in part (b)—and there is a box with j marbles for every $j = 1, 2, \ldots, n$. So from now on we assume this is the situation.

We start with the observation that a box having $m = qk + r$ marbles $(0 \le r < k)$ before a step with group size k with be left with $q + r$ marbles after that step; see Fact 6.

Lemma 1. *After the first step, with group size k, there is a number $f(n,k)$ such that the marble populations are exactly all the numbers in the range $1, 2, \ldots, f(n,k)$, and no others.*

Proof. Let $f(n,k)$ be the highest marble population in a box at the end of the step. We show by reverse induction on j (Fact 24) that there is a box with exactly j marbles, for all $j = 1, \ldots, f(n,k)$.

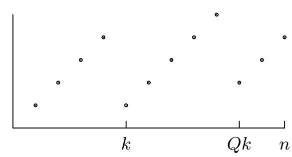
Suppose this is true for some j. Then there exist numbers m (starting population), q (quotient) and r (remainder) such that $1 \le m \le n$, $0 \le r < k$, $m = qk + r$, and $j = q + r$.

If $r > 0$, we look at the box that started off with $m-1$ marbles. If $r = 0$, we take the one that had $m - k$ marbles. Either way, we have found a box holding exactly $j - 1$ marbles at the end of the step. This completes the induction step and proves the lemma. □

This lemma effectively reduces the situation at the end of the first step to the original situation, with a smaller number of boxes (we just ignore boxes with duplicate populations). There remains to select k so the new largest population $f(n,k)$ is as low as possible.

Lemma 2. *The largest population $f(n,k)$ after the first step is given by*

$$f(n,k) = \left[\frac{n+1}{k}\right] + k - 2. \tag{1}$$



Proof. The function "population of a box at the end of the first step" grows by 1 when its first argument (the initial population) grows by 1, except when the argument is 1 less than a multiple of k, in which case the function drops by $k-2$ (see figure, where $k = 5$). Thus the maximum of the function is always achieved for an argument value that is 1 less than a multiple of k, and that is as large as possible under this condition. So let $Q = [(n+1)/k]$ be the quotient of the division of $n+1$ by k. The box that started off with $Qk - 1 = (Q-1)k + (k-1)$ marbles achieves the maximum, and its new population is $Q + k - 2 = [(n+1)/k] + k - 2$ marbles. This proves (1). □

Lemma 3. *For a given value of n, the function $f(n,k)$ achieves its minimum when $k = \left[\sqrt{n+1}\,\right] + 1$.*

Proof. To find the value of k that minimizes $f(n,k)$, we first write

$$f(n,k) = \left[\frac{n+1}{k} + k\right] - 2.$$

The function inside the brackets decreases in the interval $(0, \sqrt{n+1}\,)$ and increases in the interval $(\sqrt{n+1}, n]$. Since $[x]$ does not decrease, the function $f(n,k)$ attains its maximum either at $k = \left[\sqrt{n+1}\,\right]$ or at $k = \left[\sqrt{n+1}\,\right] + 1$.

It remains to show that we always have

$$f(n, [\sqrt{n+1}\,] + 1) \le f(n, [\sqrt{n+1}\,]).$$

Let $[\sqrt{n+1}\,] = s$. Then

$$s^2 \le n+1 < (s+1)^2. \tag{2}$$

Thanks to (1) it suffices to prove that

$$\left[\frac{n+1}{s+1}\right] < \left[\frac{n+1}{s}\right].$$

Equation (2) implies that

$$\left[\frac{n+1}{s+1}\right] \le s \quad \text{and} \quad \left[\frac{n+1}{s}\right] \ge s.$$

Therefore, it suffices to prove it is not possible for both sides to equal s. But

$$\left[\frac{n+1}{s}\right] = s \implies \frac{n+1}{s} < s+1 \implies \frac{n+1}{s+1} < s \implies \left[\frac{n+1}{s+1}\right] < s.$$

The lemma is proved. □

We are now ready to find the minimum number of steps required for any starting value of n. We simply apply repeatedly the function

$$g(n) = f(n, [\sqrt{n+1}\,] + 1),$$

which represents the maximum population after one step, with an optimal choice of k. We verify by direct computation that, after five applications,

$$g(g(g(g(g(460))))) = 1 \quad \text{but} \quad g(g(g(g(g(461))))) = 2.$$

The sequences of steps for $n = 460$ and $n = 461$ are as follows:

n	step	k	n
460			461
	1	22	
40			41
	2	7	
10			11
	3	4	
4			5
	4	3	
2			3
	5	2	
1			2

To recapitulate, we spell out the details of the argument for $n = 461$. The three lemmas say that if the boxes start with all populations from 1 through 461, then after step 1 there remain all populations from 1 through $g(461) = f(461, 22) = 41$; after step 2 there remain all populations from 1 through $g(41) = 11$; after step 3, all populations from 1 through 5; after step 4, all populations from 1 through 3; and after step 5, populations 1 and 2. Thus, for $n = 461$ it is not always possible to be left with a single marble in each box.

Remarks. 1. Instead of proving that

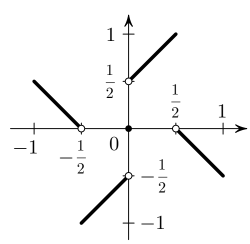
$$f(n, [\sqrt{n+1}\,] + 1) \le f(n, [\sqrt{n+1}\,]),$$

one could just check case by case.

2. This problem originated in computer programming. A text containing one or more blank spaces between words had to be processed so as to leave precisely one blank between words. The programmers solved the problem by recursively applying the operation of selecting a positive integer k and replacing every group of k consecutive blanks by a single blank.

Problem 5. (a) A possible solution is shown in the figure. The function is defined by

$$f(x) = \begin{cases} -\frac{1}{2} - x & \text{for } -1 \le x < -\frac{1}{2}, \\ x - \frac{1}{2} & \text{for } -\frac{1}{2} \le x < 0, \\ 0 & \text{for } x = 0, \\ x + \frac{1}{2} & \text{for } 0 < x \le \frac{1}{2}, \\ \frac{1}{2} - x & \text{for } \frac{1}{2} < x \le 1. \end{cases}$$



(b) We will show that a function satisfying the conditions of the problem cannot exist on the interval $(-1, 1)$. The case of the function defined on the whole real axis is analogous.

Step 1. Suppose, to the contrary, that such a function $f(x)$ exists. Its graph is mapped to itself under a clockwise $90°$ rotation: if (x, y) is a point on the graph, we have $y = f(x)$, so $f(y) = -x$ by assumption, and the point $(y, -x)$ also belongs to the graph; but this is precisely the image of (x, y) under the specified rotation.

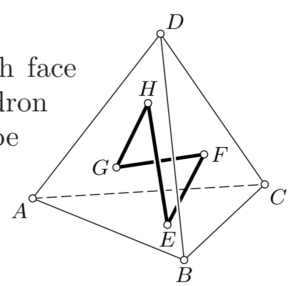
By applying the 90° rotation repeatedly we see that the graph maps to itself also under 180° and 270° rotations.

Step 2. This implies that the coordinate axes cannot intercept the graph except at the origin: any other intersection would imply three more intersections (obtained by 90°, 180° and 270° rotations), and in particular there would be two distinct intersections of the graph with the y-axis, which is impossible.

Step 3. Now consider the portion of the graph that lies within the open first quadrant $\{(x, y) : x > 0, y > 0\}$. Recalling the assumption that the graph is a union of finitely many points and line segments, we can write this intersection as $L_1 \cup L_2 \cup \cdots \cup L_n \cup P_1 \cup P_2 \cup \cdots \cup P_m$, where the L_k are line segments and the P_j are points. We may assume that the line segments L_k are pairwise disjoint and open (meaning the endpoints are excluded) and that the points P_j are distinct and do not belong to any of the line segments L_k.

For each k, let J_k be the line segment obtained from L_k by a clockwise 90° rotation. Next, for each j, let Q_j be the point obtained from P_j by the same rotation. All these points and segments lie on the open fourth quadrant, and in fact $J_1 \cup J_2 \cup \cdots \cup J_n \cup Q_1 \cup Q_2 \cup \cdots \cup Q_m$ is precisely the intersection of the graph with the fourth quadrant. (Why?)

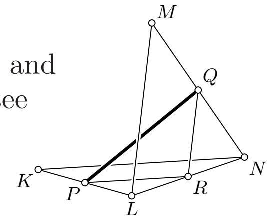
Since we already know the graph does not intersect the positive x-axis, we conclude that the intersection of the graph with the half-plane $x > 0$ consists of $2n$ line segments (the L_k and J_k) and $2m$ points (the P_j and Q_j). None of the line segments can be vertical. Thus the projections of all these line segments and points on the x-axis partition the interval $(0, 1)$ into $2n$ open intervals and $2m$ points. But it is impossible to divide an *open* interval into an even number of subintervals using an even number of points! We have reached a contradiction.



Problem 6. Clearly a shortest flight touches each face of the tetrahedron exactly once. Let the tetrahedron have vertices $ABCD$, and let the shortest flight be the space quadrilateral $EFGH$, where $E \in \triangle ABC$, $F \in \triangle BCD$, $G \in \triangle ABD$, and $H \in \triangle ACD$ (see figure). Our job is to find the smallest possible perimeter for the quadrilateral $EFGH$.

Draw the symmetry plane of the tetrahedron containing CD; note that it is perpendicular to AB. Let $E_1F_1G_1H_1$ be the reflection of $EFGH$ in this plane (so E_1 and G_1 lie on the same faces as E and G, respectively, while F_1 lies on the same face as H, and H_1 lies on the same face as F). The quadrilaterals $EFGH$ and $E_1F_1G_1H_1$ have the same perimeter.

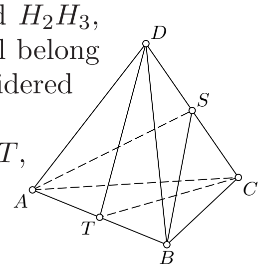
Lemma. *In any space quadrilateral, the distance between the midpoints of two opposite edges is less than or equal to the mean of the lengths of the remaining edges.*



Proof. Let $KLMN$ be the quadrilateral and let P and Q be the midpoints of KL and MN, respectively (see figure). Denote by R the midpoint of the diagonal LN. We have $PR = \frac{1}{2}KN$ and $RQ = \frac{1}{2}LM$. Hence $PQ \le PR + RQ = \frac{1}{2}(KN + LM)$. $\square$

Denote the midpoints of the segments EE_1, FH_1, GG_1, and HF_1 by E_2, F_2, G_2, and H_2, respectively. These points also lie on the faces of the tetrahedron. By the lemma, the perimeter of $E_2F_2G_2H_2$ is no greater than that of $EFGH$. Moreover, E_2 and G_2 lie on the symmetry plane of the tetrahedron containing CD; that is, they lie respectively on the medians CT and DT of the faces ABC and ABD, where T is the midpoint of AB.

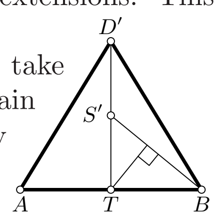
Next we repeat this symmetrization operation for another symmetry plane. That is, we reflect $E_2F_2G_2H_2$ in the symmetry plane of $ABCD$ that contains AB, obtaining a quadrilateral $E_3F_3G_3H_3$, and then we take the midpoints of the segments E_2G_3, F_2F_3, G_2E_3, and H_2H_3, obtaining a quadrilateral $E_4F_4G_4H_4$, whose vertices all belong to one of the two planes of symmetry of $ABCD$ considered so far (one through AB and the other through CD).



Specifically, the vertices E_4 and G_4 lie on CT and DT, while F_4 and H_4 lie on the medians AS and BS of the faces BCD and ACD, where S is the midpoint of CD. (See figure on the right.)

Again, the perimeter of $E_4F_4G_4H_4$ is no longer than that of $E_2F_2G_2H_2$, which as we know is no longer than that of $EFGH$. Hence, the perimeter of $EFGH$ is at least $4d$, where d is the distance between CT and BS.

It remains to construct a path of length $4d$ and find d. Let the common perpendicular to CT and BS intersect CT at E_0 and BS at F_0. Let G_0 be the reflection of E_0 in the plane ABS. It follows from symmetry that F_0G_0 is the common perpendicular to BS and DT. Similarly we construct the point H_0 such that G_0H_0 is the common perpendicular to DT and AS and H_0E_0 is the common perpendiculars to AS and CT. The perimeter of $E_0F_0G_0H_0$ is $4d$. We also have to prove that the bases of our common perpendiculars lie on the faces of the tetrahedron, rather than on their extensions. This will be checked below (we still have to calculate d).



Draw the plane through AB perpendicular to CT and take the projection of the tetrahedron on this plane. We obtain triangle ABD', in which $AB = a$ and $D'T = a\sqrt{2/3}$, by the formula for the length of the altitude of the regular tetrahedron. (See figure on the right.)

The projection sends S to S', the midpoint of $D'T$. Hence, d equals the distance between T and the line BS', because the common perpendicular is parallel to the plane of projection. It is also obvious that the base of the perpendicular dropped from T onto the line BS' lies on the segment BS' and not on its extension; hence F_0 lies on the segment BS. We similarly prove that the remaining vertices of the quadrilateral lie on the medians, not on their extensions.

In the right triangle BTS', the legs $BT = \frac{1}{2}a$ and $TS' = \frac{1}{2}a\sqrt{\frac{2}{3}}$ are known. Hence

$$BS' = \tfrac{1}{2}a\sqrt{\tfrac{5}{3}}, \qquad d = \frac{BT \cdot TS'}{BS'} = \frac{a}{\sqrt{10}}.$$

Remarks. 1. The solution implies that there are three suitable paths. (Why?)

2. An analogous problem on the plane is known: A beetle crawls inside a triangle with sides a, b, c. What is the shortest length of a path that visits each side and returns to the initial point?

   It the case of an acute triangle the answer is the path joining the bases of the altitudes; this is known as *Fagnano's problem.* (See [306], Chapter 4, § 5.) For a right or obtuse triangle the path degenerates into an altitude traveled twice; see Problem 70 in [305].