Chapter 1

The First Session: Narrative and Reflections

(The first half of this chapter is based on my article, "Mathematics for Little Ones," published in *The Journal of Mathematical Behavior*, vol. 11, #2, 1992.)

A session in action

Our circle has four members: my son Dima and his three pals Gene, Pete, and Andy. Dima is the youngest, at 3 years, 10 months old; the eldest is Andy, who's almost five.

We sit down around a coffee table. I am nervous: will I be able to handle them? To begin with, I tell the children that we are going to do some math. To increase my authority I add that math is the most interesting science in the world. Of course, this is immediately followed by the question, "What is science?"

I have to explain, "Science is when you think a lot."

Andy is somewhat disappointed, "I thought there would be tricks." At home he has been told that I would be playing math and performing tricks.

"We'll have tricks too." I cut short the introduction and get down to business.

The first puzzle: I put eight buttons on the table. Without waiting for my instructions the boys rush to count them; apparently, despite their young age, they have already been indoctrinated with the idea that "math" means "counting." When they calm down I can explain the puzzle:

"Now put on the table as many coins."

We put 8 buttons and 8 coins into two parallel, equally spaced rows.

"Which are there more of, buttons or coins?"

The kids are puzzled, they hesitate before answering:

"Neither is more."

"That means equal," I say. "Now watch what I am going to do." And I move the coins apart making their row longer.

"And now, what are there more of?"

"Coins!" shout the kids together.

I ask Pete to count the buttons. We have already counted them four times, but Pete is not in the least surprised and sets off to count them for the fifth time.

"Eight."

I ask Dima to count the coins. He counts and announces the result:

"Eight."

"Also eight?" I emphasize my words. "So as many coins as buttons?"

"No, there are more coins!," the boys insist with conviction.

To tell the truth, I knew beforehand they would answer in this way. This puzzle is one in the series of tests used in experiments by the great Swiss psychologist Jean Piaget (I will speak in more detail about the "Piaget phenomena" in the next section). Piaget showed that young children do not understand things that seem self-evident to adults: if several objects are moved or rearranged, that does not alter their number. Although I knew beforehand what the kids would say, I had not prepared any reasonable response! What would you have done if you had been in my shoes? What would you have told the kids?

Unfortunately the most frequent reaction of adults is to go out of their way to "explain." "How can it be?," might say an adult with feigned surprise. "More coins? But we have not added any! We've just moved them apart, that's all! Before there were as many buttons as coins, you said that yourself. So, how can there be more coins all of a sudden?" More loudly, "Of course there are as many buttons as coins!"

No matter how you try, this teaching strategy will get you nowhere. To be more precise, it will lead to an impasse. First, don't think that your logic will convince any child. Mastering logic comes much later than the law of number conservation. Until then, logic will not seem convincing to that age group. What is convincing is your intonation, which will show the kid that once again he did something wrong. Kids do not give up easily; their common sense is hard to beat down. But if you exercise sufficient pressure you get your way: they will cease trusting their intelligence and experience and will try instead to guess what the adult wants them to say. Adults in general demand lots of weird things: you can't draw on a wall; you have to go to bed in the middle of a game; you must not ask "When will Uncle George go home?" The same thing is happening now: I see perfectly well that there are more coins than buttons but for some reason I have to say they are as many.

The acceptance of mathematics as a ritual, where you have to pronounce certain incantations in a certain order, starts in elementary school, and sometimes survives through high school and beyond.

So what to do? Should you avoid these questions if you cannot talk about the answers?

On the contrary, questions can and must be asked. It's also very useful to exchange opinions. "And you, Gene, you think the same? You, Pete? Why? How many more coins are there?" You might even give your own point of view, but carefully, accompanied with all kinds of "I think," "probably," or "maybe." In other words, you should use all your adult authority not to impose "correct" answers but to convince a child of the importance of his own reasoning and intellectual efforts. And it is even more interesting to make him discover contradictions in his point of view.

"How many coins must we take away so that there again will be an equal number?"

"Two."

We take away two coins and re-count; we have 8 buttons and 6 coins.

"What is there more now?"

"Now it's equal."

Fine. Again I move the remaining coins apart and ask the same question. Now it turns out that 6 coins are more than 8 buttons.

"Why so?"

"Because you moved them apart."

We take away two more coins; then two more. What we have at last is shown in Figure 2.

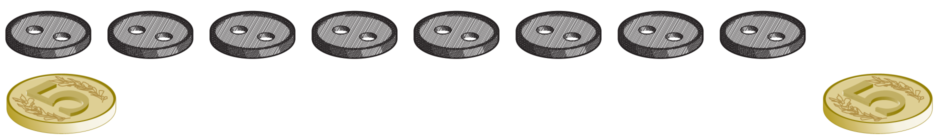

Figure 2. There are 8 buttons in the upper row and 2 coins in the lower one. What is there more of, buttons or coins?

At this moment the boys start a heated discussion. Some still think there are more coins; others suddenly "see" that there are more buttons. It's time to stop and pass on to another puzzle. Let them think for themselves.

> *I was among those who said there were more coins. At first I agreed with the others and then just repeated without thinking. It seemed correct all the previous times (at least Dad did not say otherwise), so I did not see why this time I should change my mind. — Dima*

I did not have all these ideas at once, so in my story I've gotten a little ahead of events. What I am describing now are my later reflections and future sessions. We repeated the same puzzle in different guises. For instance, once we had two armies none of which could win because they had equal number of soldiers. Then soldiers of one army moved apart, their number increased and they started to win. The soldiers of the other army, having seen that, moved even further apart, etc. (Use your own imagination to finish the story.) We also had a puzzle where the fox and the cat wanted to swindle poor Pinocchio moving apart five golden coins and saying that their number grew. I learned not to expect easy victories. No matter what I do, the kids won't master the law of number conservation for another two or three years. Nor should they! Premature instruction is no more beneficial than premature birth. Every vegetable has its season; we must not run ahead of the natural course of things, including intellectual development. (I realize that this may seem strident. But I hope to persuade you of this point of view during the course of the book.)

However, as I have said earlier, all these reflections came later. During our first session, my intuition told me to stay away from explaining things, so I merely moved on to a new puzzle.

I took six matches and made different figures. After I have composed a new figure, I asked the kids, in turn, to count the matches. And each time there are six. — No, it won't do. My style has become too dry and too formal. Let's directly observe the kids and see what actually transpired.

Each result is welcomed by a burst of laughter and delight. Andy and Gene shout that there will always be six! Dima, rather impolitely, tries to wriggle matches out of my hands: he wants to make his own fanciful figure. Pete, on the other hand, is exquisitely polite and asks me whether I can give him several matches. In a few moments, their delight turns into chaos. I have to stop them, to listen attentively to what they shout ("Why do you think there will always be six?") and not to miss new turns of their reasoning: Dima has just made a three-dimensional figure — a "well" (Figure 3).

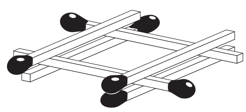

Figure 3. A "well" made of six matches.

I invite everybody to look at it. This time even Andy and Gene are not that sure we will have six matches. It's difficult to count the matches, the well is not very solid, it collapses, we restore it, count matches, it collapses again... Finally, Dima arrives at seven. The boys are a little puzzled, but not much surprised. So seven it is, even though it's a little strange.

I must reiterate — once more — that my task as a teacher is not to dictate "the truth" to the kids, but to stimulate their curiosity. The best I hope to achieve is that in a few days (or months), one of my boys will build a well of matches and count them, because he wants to investigate. Modest — but independent — research! And if this does not happen, I'll hope that it will happen another time, with another puzzle. (Indeed, this did happen several times.) Anyway, I only say something like "how interesting!" and "amazing!" in the hope that this way it will stick in their memory.

Children have remarkable memories. I can't help telling a story that occurred much later: We were discussing the properties of three figures made of cardboard (Figure 4). First, each figure has four angles. So we can call each of them a quadrilateral. That means we have three quadrilaterals. But all the angles of two of the shapes are right. That's why they are called rectangles. One of the rectangles is special: all its sides are equal. This

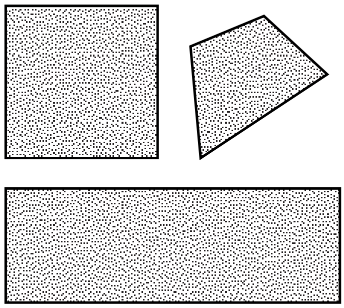

Figure 4. How many squares are there? How many rectangles? How many quadrilaterals? These questions are not easy even for adults.

is called a square. We can say that a square has three names: it may be called a square, a rectangle and a quadrilateral, and each of these names will be correct. The boys accept this information not without argument. They stubbornly try to think in terms of disjoint classes. As to their explanations, they make me think they have not yet comprehended the great law "the whole is bigger than its part." Ten minutes ago they were disputing whether dads and grandfathers were men and whether men were people. Now they refuse to call a square a rectangle: it is either one or the other. I intensively agitate for the rightful inclusion of squares as rectangles. By and by my lobbying starts to make headway. We sum up once more:

"How many squares do we have?"

"One."

"Rectangles?"

"Two."

"And quadrilaterals?"

"Three."

Everything seems to be OK, so now I can pass on to the last question (I've already mentioned it in the introduction):

"And in general, in life, what is there more of: squares or quadrilaterals?"

"Squares!" answer the boys together without a shade of hesitation.

"Because they are easier to cut out," explains Dima.

"Because there are many of them in houses, on roofs, on chimneys," explains Gene.

This is just the set-up of the story. The punchline came a year and a half later and without any prompting on my part. In summer, during a walk in the forest Dima suddenly told me, "Remember, Dad, you once gave us a puzzle on squares and quadrilaterals, what is there more of. Well, I think we did not give you the right answer then. In fact, there are more quadrilaterals."

And he rather intelligently explained why. Since then I've adhered to the principle that questions are more important than answers.

For decades, psychologists have been carrying out experiments attempting to teach children certain basic mathematical laws. Here is an example: Initially, a group of kids is tested to see whether they understand that if we squish or roll or otherwise modify the shape of a piece of modeling clay, the quantity of clay will not change. Those who do not understand are divided into two groups. One is left alone, as a control group, while psychologists start to work with the other trying to teach them the law of conservation of matter; they weigh, compare, show, and explain. Two weeks later they test the participants of both groups to see what the kids have learned. Most often than not the progress in both groups proves to be insignificant and more or less equal. As a rule, psychologists are perplexed why the kids who were taught so patiently never learned much. When I was reading about this, what startled me was the opposite: why the kids of the control group, whom nobody was teaching, managed to make some progress. After a few years of dealing with kids I think I know the answer: it happened because they were also asked questions.

Let us however return to our first session. Our next puzzle was yet another version of the same law of number conservation. Those six matches are still on the table. I put them in a row and ask to put a button opposite each of them (Figure 5).

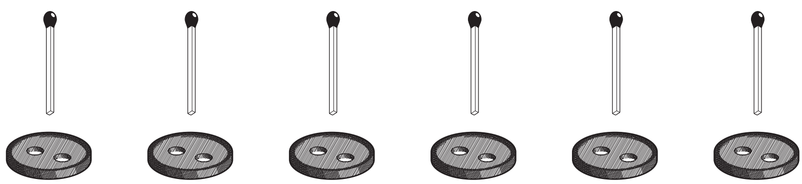

Figure 5. As many matches as buttons.

Then comes the standard question, “What is there more of, matches or buttons?”

“Equal.”

I sum up, “That is, as many buttons as matches.”

I take away all the buttons, hide them in my fist and ask how many they are. It’s typical that none of the boys makes an attempt to count the matches. Why should they? The question was about buttons, so you have to count buttons. Dima, being on the most intimate terms with me, tries to open my fist, and others ask, bewildered, “But how can we count them?”

I laugh, “Of course you can’t, they are hidden, but try to guess.”

They toss out a flurry of wild guesses. Each shouts a different guess, only Gene’s being the right one. I try to listen to him and to ask why but at this moment he retreats. His problem in general is that he is timid. When all the boys holler, he is the one who shouts the correct answer more often than others do; but the moment I calm the others and address him directly, he is embarrassed and falls silent. My problem with Andy is of a different sort. He is a very goal-oriented boy yet apparently lacks motivation at our lessons. Next time, I camouflaged the same puzzle as a war scene: instead of matches and buttons, we had soldiers with guns, then the soldiers went away leaving their guns behind, and the spy had to learn the number of soldiers, and it was he who first got the idea to count guns. He also loves games where one can win. But I don't always have enough imagination to package my puzzles appropriately, especially since it's not that important for the other boys. As for Dima, he does not fancy solving other people's puzzles but prefers to invent his own. Eventually I learn how to handle him: I say something like, "Now think of a puzzle where..." and then give my own formulation. Also, his solutions are often oddly fanciful (as the next example vividly illustrates) and it is difficult to hold him within the limits of common sense. Pete too has his idiosyncrasies. How can I cope, alone, against them? Goodness, with only four pupils I am unable to provide an individual approach. What about a schoolteacher with forty pupils in a classroom? A teacher is often compared to an orchestra conductor. But in my own eyes I rather resemble a juggler who in a minute will drop all his balls on the floor. While I am trying to speak to Gene, Dima has already pulled out cards for the next assignment ("Odd One Out") and asks, "Dad, is this the next puzzle?"

The other two wrest the cards from his hands and mercilessly crumple them without the slightest regard to the overnight parental labor. Gene is no longer listening to me but is looking sideways at them. I open my fist; we briefly check the number of the buttons and pass on to the next question.

The rules of Odd One Out are well-known. Children receive four cards with the pictures of, say, a rabbit, a squirrel, a hedgehog[1], and a suitcase. They must say which one is odd. It is amusing to watch how the kids almost always answer correctly but often cannot explain why.

"It's a suitcase which is odd."

"Why?"

"Because it is not a rabbit, a squirrel or a hedgehog."

"Really? And I think that it is the rabbit which is odd. Because it is not a squirrel, a hedgehog or a suitcase."

The boys give me perplexed looks, but insist:

"No, it's a suitcase which is odd!"

I ask whether it will be possible to find a single word to name the three not-odd objects: rabbit, squirrel, hedgehog. Pete, who is well ahead of others in his vocabulary is the first to find the word: "animals." Later he was often helpful in similar situations.

(Once I was invited to give a lesson for the children of roughly the same age, four or five years, whom I never met before. I put my favorite cards with a rabbit, a squirrel, a hedgehog and a suitcase on the table and asked which was an odd one. The kids looked at me, horrified, then one of them got up his courage and whispered: "They're equally...." Aha, thought I, someone has already put them through their paces!)

Sometimes I propose puzzles with ambiguous solutions, for example: sparrow, bee, snail, plane. The odd object could be a plane (not alive) or a

[1] *P. Z.'s Note:* Hedgehogs are ubiquitous in Russian children's literature.

snail (cannot fly). Figure 6 shows a puzzle in which each of the objects could be the odd one. This completely alters the nature of the problem. Now I name the odd objects one at a time, while the kids have to explain why they could be odd. Thus I hope to inspire them with the mathematically important idea that we need good explanations as much as correct answers. In other words, not only correct statements but also their proofs.

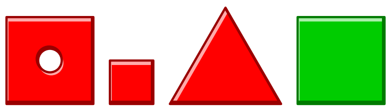

Figure 6. Instead of looking for an odd object, the kids must nominate them as odd, one-by-one, and then explain why each one is odd.

Odd One Out is very handy for teaching children to guess regularities (totally ignored by school mathematics). Sometimes it is more convenient to use eight pictures that can be separated into two equal groups according to the chosen categories; this approach was used by the Soviet computer scientist M. Bongard in his classic book *Pattern Recognition*. Sadly, you will readily agree that eight pictures are twice as many as four. How to get them? Usually Alla drew pictures for our sessions. I cannot draw at all, while she once attended art school.

Intersecting classes can be logically challenging. For example, suppose five pictures must be broken into two equal groups, each containing three pictures. One picture will belong to both groups. Example: ball, tire, rubber boots, coat, hat. Three objects are made of rubber (ball, tire, rubber boots) and three others are clothes (rubber boots, coat, hat). The common object is rubber boots. But it's difficult, conceptually, to physically divide five pictures into two groups of three each, since we can't just tear one picture in half! We use the pictorial technique of Venn diagrams to produce two intersecting circles, the common object being placed inside the intersection (see Figure 7 for a similar puzzle).

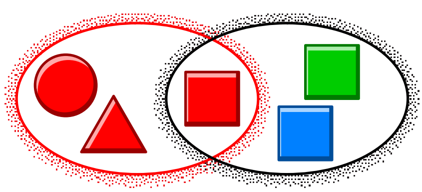

Figure 7. Here are two sets, each containing three objects. One set consists of three red objects, the other, of three squares. The red square is common to both; mathematicians say that it belongs to the intersection of these sets.

These sorts of problems clearly were a problem for Dima (or was *he* the real problem?). "This one is an uncle but it looks like a lady," he would say, pointing to a picture of a bearded old man, and place it in the women's pile. He would say that a tire was a piece of clothes because you could put it around your waist. When contradicted, he would explain that it was a car's clothes.

Some will say, "What a creative, off-beat child!" Off-beat, sure. But creative? A truly creative person can propose an unexpected, non-standard solution within the bounds of the problem. Making a well of six matches is creative. But classifying a bearded old man as a woman or insisting on a tire being a car's clothes is not. Dima often displays the first component, off-beat inventiveness, but lacks the ability to remain within the problem formulation or at least in its vicinity. It would be nice to nurture the latter without suppressing the former. But how?

We end the session with a geometry puzzle. I get out a colorful mosaic pegboard that I bought in the pre-circle days, so — alas! — I have just one. It's a rectangular array of holes that can be filled with pegs that come in five vivid, appealing colors (Figure 8).

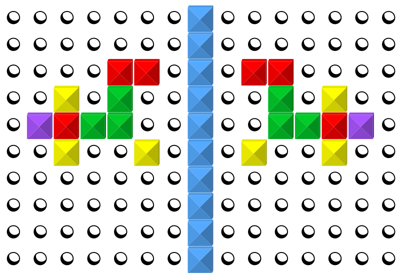

Figure 8. A mosaic pegboard. The vertical line of pegs in the middle is a "mirror", or symmetry axis. The figure on the left is made by the teacher; the pupil must make a symmetric one on the right.

Our puzzle involves symmetry. First I construct an axis, a one-color vertical line in the middle of the array and call it a "mirror." We will make figures that "look in the mirror." I construct different small figures on the one side of the mirror while the boys have to construct symmetrical figures on the other side of it. I try to vary all possible parameters: color, size, position. In later sessions, I will also change the direction of the axis: it will become horizontal and then diagonal. We check our solutions with real mirrors: is what we see *behind* the mirror the same as what we see *in* the mirror?

The boys deal with with this puzzle easily, making almost no mistakes. I can't understand why this subject (axial symmetry) causes such great difficulties in middle school.[1] We devoted many sessions to this rich and beautiful topic. We admired and drew symmetrical patterns; made symmetrical ink-blots by folding a sheet of paper into two; cut out symmetrical Christmas snowflakes; looked for "errors" in drawings containing deliberate violations of symmetry; sorted out cards containing symmetric and non-symmetric figures, etc. Other types of symmetry, (e.g., rotational symmetry) proved more difficult, but axial symmetry was a great success.

The pegboard soon became one of my favorite tools. It's a great resource for puzzles in geometry, combinatorics, logic, and discovering patterns. It even gave me a precious lesson about "what is more important for children:" The boys enjoyed the sessions and sometimes, after I called it a day, would ask me to go on. I felt smug until I noticed that they asked me to continue only when we played with the pegboard. I decided to test this: We didn't

[1]Specifically, the Russian 6th year, roughly equivalent at that time to 7th or 8th grade in the United States.

use the pegboard at our next session, and when I said the session was over, the boys left without further comment. I felt grave doubts: The pegboard is indeed a beautiful toy; no wonder the boys enjoy playing with it. But my mathematics is just an imposition interfering with this toy. At the next session, I decided to try the ultimate test. Once again, we use the pegboard, and once again, the boys do not want to stop. I say, "We'll end the lesson, but you can stay and play with pegboards." I receive an unanimous yell of indignation, and Pete sums up their mood with, "No-o-o, we want puzzles!"

Thus I have come to realize that kids must have fun in both intellectual and aesthetic ways. If one is missing, there is no joy. A Christmas tree without ornaments is as unappealing as are mere ornaments without a tree. The joy is in their union. I hope that one day, when my pupils are engaged in more abstract and "intellectual" mathematics, they will enjoy it more than their peers, because abstract notions and images will subconsciously blend with a "Christmas tree feeling;" the memories of brightly colored puzzles from their childhood.

By now, each kid has completed two puzzles and it's time to stop, but they want more. Suddenly I get an idea, and say, "Why don't *you* give me puzzles and *I* will do them." The kids are delighted. They make figures with a renewed enthusiasm, and I construct symmetric ones. I work industriously. Then I get another idea, and start deliberately making mistakes. Pete, ecstatic, is the first to notice. They've gotten a second wind. They watch my hands intently and greet each mistake with savage war cries. But it is really time to stop. I put the pegboard aside, thank everybody and say that the session is over.

"And when there will be tricks?" remembers Andy.

"Why, Andy! It was you who did the tricks! I hid the buttons in my fist but you could count them!"

As a matter of fact, it was not him, it was Gene who guessed how to count the buttons but Andy looks satisfied. We get up. I glance at my watch. Is it possible that our lesson only lasted 25 minutes? The boys go away and I stay to order in my thoughts, invent new puzzles, and think up new techniques and strategies. And also, to cut, glue, and paint; i.e., make the "visual aids" and "manipulatives." I have just a week to get ready for the next session.

Piaget's phenomena: reality or illusion?

Since I frequently refer to Piaget's phenomena, let's say a few words about them. The great Swiss psychologist Jean Piaget was undoubtedly one of the most influential people in his field during the twentieth centrury. During his long life (1896-1980) he wrote about 50 books and some 500 articles. In 1976 he celebrated a very unusual anniversary: his 80th birthday and his 70th year as a scientist. No kidding! He published his first paper at the age of 11: an observation of an albino sparrow in a park.

As a young student, Piaget became interested in malacology, the science of mollusks, and soon became an acknowledged expert. On the basis of his publications he was offered — by correspondence — a prestigious curatorship of a mollusk collection in the Geneva Museum of Natural History. He had to confess that he was still a high school student. By the age of 20 he was a malacologist of world renown. At this point, his interests abruptly changed to child psychology. By 30, he had written five famous monographs and was a world leader of the field.[1]

Of all of his vast work, the most famous are the so-called "Piaget's phenomena" that I've mentioned earlier. A young child does not understand that if we move a few objects (stones, blocks), their number does not change. Therefore the notion of a number remains inaccessible to him though he may know how to "count up to one hundred." Eventually he will understand the above-mentioned law of "number conservation." But we still must wait about a year and a half or even two years before he understands the same idea, but with continuous quantities: if we roll a ball of modeling clay into a hot-dog, the quantity of clay won't change; neither will the quantity of water change if we pour it from a glass into a tub. Likewise, if we have two equal quantities, and we take a bit from one quantity and a bigger bit from the other, then the first will have more material left than the second. It is hard to believe that kids do not understand what is so obvious to us!

Remarkably, Piaget's discoveries involve simple observation and do not require sophisticated equipment like rockets or lasers. Plato or Euclid could have made the same discovery. But it never occurred to them. What was needed was a person interested in cognition, able to ask the right questions, endowed with an outstanding power of observation and, willing to carry out extensive experimentations.

It is interesting, however, that Piaget's findings were strongly opposed by some members of the scientific community. Even now, decades later, some people say, "When we ask a child we use the words 'more' and 'less', don't we? Who explained to him what these words mean? The child does not understand them the way we do, it's as simple as that." This idea was best formulated by a logician friend of mine: "You did not give them the definition of the word 'more.' That's why they understand it in their way. They think that 'more' means 'a longer row.'"

How could I refute this? Indeed, I did not give them any definitions. But what kind of definition? Should I have said that there exists a bijection[2] between one set and a proper subset of the other set? That would hardly have helped: how can we be sure that if such a bijection exists once, it will exist later? I argue with my logician friend, but without conviction. I tell him about the experiments with children, how they weighed pieces of clay

[1] The best overview of his theory can be found, to my mind, in the book by John H. Flavell *The Developmental Psychology of Jean Piaget* (D. van Nostrand, Princeton, 1963).

[2] The technical mathematical term denoting a one-to-one correpondence; if there is a bijection between two sets then they have the same number of elements.

before and after rolling them into hot-dogs... but, so what? A child has no idea what a balance is and what it measures.

We can't escape from this vicious circle of argument. No matter what we tell the child, no matter how we formulate our questions, there will always be an intermediary link, a "signal-carrier", be it words, a balance, or calculations. And we can always blame this "interface", this "exchange protocol," claiming it to be not sufficiently precise; and therefore our interpretations differ from the child's. True, we can ask our opponents, "Why then, by the age of seven, the child answers our questions correctly though he never had any definitions?" But this won't do: in a scientific discussion we cannot ask, "Why then...?" An opponent is not obliged to prove anything — this is the burden of the proponent of the theory — of course, insofar as a proof in psychology is at all possible.

Without getting bogged down in the philosophical complexities of this, I'll share my personal point of view. After years of working with kids, I did not need any proof. I knew that Piaget was right. I observed these phenomena so frequently and under such different circumstances that I do not need to be convinced. I remember once we had visitors and were one chair short, and Dima, then aged 3, tried to make the guests change their seats to accommodate everyone. Each time there was one chair short. It was enough to see his bewildered expression to understand that it had nothing to do with the semantics of the word "more." (But I would certainly not have noticed this had it not been for Piaget.)

Psychologists have spent much time and ingenuity trying to teach children these conservation laws (or, as our opponents might say, to explain to them the exact meaning of their questions). As a rule they had no success.[1] Here is my favorite story: They selected, from a large group of children, those who seemed to have "understood everything." At least they gave correct answers to the psychologists' questions: "There is as much modeling clay as before because we did not add nor take away anything. We only changed the shape." Then the researchers took another step and they tried to *unteach* these children. The child gives the correct answer; together they weigh the piece of clay, but it becomes lighter! What happened? The mischievous experimenter secretly pinched away a small bit. And it turned out that the children who easily learned the law now unlearned it just as easily. They started to say that now there was less clay because we rolled it into a hot-dog. But the children who mastered the law of conservation before this experiment *on their own* could not be untaught. They said, "Perhaps we have dropped a small bit and have not noticed it."

OK, if it is so hard to teach kids the notion of a number, what am I trying to do? What is the point of my lessons? I said it many times and I am going to say it again: the meaning of the lessons is the lessons

[1] On p. 31 I'll describe a case when very modest success was achieved.

themselves. Because they are fun. Because it's fun to ask questions and look for the answers. It's a way of life.

. . .

I'll end with a few stories. The first is from my own childhood, when I was about five. We lived in Vitebsk (in Belarus, then a republic of the Soviet Union). In our courtyard, there lived an old man who liked to talk to children. I was considered a "smart boy" who knew how to count. So once he asked me to multiply three by five. I knew already that "to multiply means to add the same number as many times as necessary." So I launched myself into this perilous adventure. Three and three makes six, easy so far. Six plus three is nine; a bit more difficult, though the hardest thing is not adding but remembering how many times I did it. Next comes the most difficult step: $9 + 3$. First, it's above coming over ten; second, I must not forget how many times I have already added. Finally, at the limit of my intellectual capacities, I added 12 and 3 and said, "Fifteen."

"Good!" said the old man. "How did you do it?"

I told him.

"Why so complicated?," he asked. "You could have simply added five plus five plus five."

I was overwhelmed, and at the same time completely lost. I can add $5+5+5$ hands down! After all, $5+5 = 10$, which is trivial! And $10+5 = 15$ is also trivial! The most amazing is that you also get 15. But why!?

This mystery stuck in my mind for a long time. I was looking for an explanation, but did not find any. At school I learned that in the 6th year of school we would have algebra, and there would be formulas. Kids do not often have an idea to leaf through a textbook for the next grade. So I was patiently waiting for the 6th year, hoping to receive the long-awaited illumination. In the 6th year, I wrote the formula $ab = ba$, looked at it dumbly for some time, but no illumination came. Three years later, at the age of 15, I entered the famous Kolmogorov mathematical boarding school. The curriculum was quite advanced and soon we were learning about groups, fields, and rings. "What an idiot I was," I thought. "That was just an axiom, it is called commutativity. One doesn't prove axioms."

Eventually, I got wiser. I understood that the commutativity axiom was not invented as a whim, but because this property truly holds for the multiplication of natural numbers.

(Note that exponentiation, that is, "repetitive multiplication," is *not* commutative. If you multiply 5 by itself three times and then multiply 3 by itself 5 times, you will get two different results. On the other hand, with multiplication, i.e., "repetitive addition," the result *will* be the same.)

I don't remember when and why I finally understood that it was simply a matter of counting the same number of objects in two different ways. We can arrange stones in three rows with five stones in each row; or we can arrange them as five rows, each containing three stones, depending on what you consider to be a row (Figure 9). Thus, if you count the same number of objects in various ways, you will always get the same result! But it also means this property is not so self-evident, for otherwise it would not have taken so many years and such intense intellectual effort.

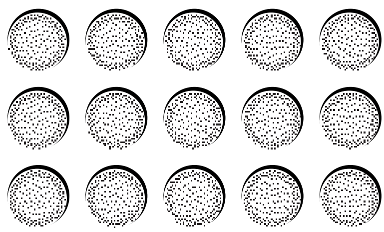

Figure 9. You see 3 horizontal rows with 5 circles in each, i.e., $5 \cdot 3$. But we can also say there are 5 vertical rows with 3 circles in each, i.e., $3 \cdot 5$. If you believe that no matter how you count you will always get the same result, the unavoidable conclusion will be that $5 \cdot 3 = 3 \cdot 5$.

The last illustration is a dialogue overheard between an elderly husband and wife. The wife plans to cook scrambled eggs but her regular pan is dirty and she does not fancy washing it right now.

"Listen," she says to her husband, "the big pan is dirty."

The husband answers, slightly irritated at being disturbed, "Well, make it in the small one."

"I am afraid there won't be enough."

Husband, thinking it over and shrugging his shoulders: "Then wash the big one."

But what about the law of conservation? Clearly it depends on the situation; suppose you gave the same couple a formal "intelligence test": if we pour beaten eggs from one pan to the other, will their quantity (a) increase; (b) decrease; (c) remain the same; (d) depend on the size of the pan? I do not doubt a minute they would answer it correctly. Which gives rise to questions that I find hard even to formulate. First, they concern the relations between formal learning and real comprehension. Second, are we in our everyday life guided by "correct reasoning" or by what is "visually obvious", and to what extent? (Have you noticed the amount of quotation marks (and parentheses!) in these last sentences? It's because I don't know how to express my idea in a brief and clear manner.)

Why read psychology books?

It is hard to convince mathematicians that psychology books are worth reading. Everything about psychology causes discomfort: terminology, the level of provability, the type of problem formulation. For example, here's an abortive conversation with a student about a series of experiments.

"For example, we'd like to know whether a two-month old baby is capable of learning."

The student shrugged and said, "Isn't it obvious? Had they asked me, I would have told them."

How can you respond to that? Indeed, it is “obvious.” A French acquaintance reacted in the same way when he learned that a Nobel Prize in economics went to someone who proved that economic behavior was neither rational nor logical.

“He could have asked my doorman,” remarked the Frenchman.

I felt that my student was wrong but it is only later that I found a refutation. Let’s switch subjects, and ask if physical laws are the same at different places and at different moments of time. The answer *seems* to be as obvious as before. Any philosopher will tell you this must be so, otherwise they are not physical *laws*. He is correct, no doubt. But what will a physicist say? His situation is more complicated: he has to deal not with general statements but with concrete laws, say, Maxwell’s equations. He will also have to concretize the vague sentence about time and space by explaining what exactly changes and in what way when we pass from one system of coordinates to another. Try to develop this idea and you will first discover the Lorentz transformations and then Einstein’s relativity theory. And this is only the first step; Maxwell’s equations describe electromagnetic forces but there are also weak, strong, and gravitational forces. For centuries, physicists have strived to give concrete form to the “obvious” philosophical principle of the temporal and spatial invariance of physical laws, and they are by no means finished with this task. Hopefully it will lead to a “unified field theory” (sometimes called half-jokingly “the theory of everything”).

To sum up: the core of the problem is to speak not in general philosophical terms but to ask concrete questions and deal with observable and verifiable phenomena. Of course, we can’t really use formulas and equations with psychology. Nevertheless, instead of asking whether a baby is capable of learning, we can ask a specific question; e.g., can a two-month baby memorize the four-bit sequence 0011? Compared to the original question, this one seems rather thin, but getting an empirical answer, even for this simple question, is not easy.

First of all, how can we know that the baby really *learned* the transmitted sequence 0011? Perhaps we can check if, to achieve some goal, the baby turns its head twice to the left and then twice to the right. But it is much more difficult to find a goal for the baby, and to make the baby want to achieve this goal. What can it be interested in? With animal experimentation the solution is simple: starvation (sorry to say that!). The weights of some experimental animals are reduced to 80% of normal, so that, while searching for food, they will be resourceful. Thank God, researchers don’t treat babies in this way! What then?

In psychology, breakthroughs rarely are found by directly investigating a question, but instead through unexpected detours. Indeed, the investigation of the above problem revealed new truths. Researchers used many “attractors:” bright rattles, melodies, colored lights. Finally they found that a simple electric bulb was sufficient. As for the stimulus, it was just *the opportunity to learn!*

Here's what happened. A baby discovers quite by chance that when it turns the head to the left, a bulb lights up. Several trials suffice to support this observation, then the baby calms down, checking from time to time if the "rule" still works. At some point, it discovers that the bulb no longer lights up. The baby tries to actively look for the reason until it makes a new discovery: to make the bulb light up it now has to turn its head once to the right, and once to the left. A new cycle of verification is followed by a new period of complacency. Then the baby discovers once again that the bulb does not respond to the "rule." A new active search follows, a new solution is found, and so on, until 0011 is "learned."[1]

Thus, the principal stimulus for learning is not a prize, not a symbolic candy after school, but school itself, the possibility to learn new things. Our task is not to smother it, not to suppress this inherent urge for learning; our task is also to create a rich environment for the infant to sustain his/her interest for the world. Here, too, psychology can give us quite unexpected insights. I quote below from the book by V. S. Rotenberg and V. V. Arshavsky, *Search Activity and Adaptation* (Moscow, Nauka, 1984):

> American researchers Jones, Nation and Massad tested four groups of subjects. At the initial stage of research the first group got problems, none of which it could solve (0% of success); the second group got problems and could solve all of them (100% of success); subjects belonging to the third group were successful in solving every other problem (50% of success). After that the subjects of the above three groups and of the fourth one (the control group) were given problems which do not in principle have a solution, i.e., the researchers tried to develop the so-called 'learned helplessness'. At the final stage all the subjects got solvable problems of average difficulty. In this way the researchers wanted to learn the efficiency of the previous stage. What came out was that only the subjects of the third group had developed immunity against the learned helplessness. They were the best in solving the final stage problems. No meaningful difference was observed between the first, the second, and the control group. The most interesting result here seems to be that both 100% failure and 100% success were similarly inefficient in increasing the resilience of the subjects in the face of the later failure.

Similar results have been obtained in experiments with children, with puppies, and with baby rats. It gives you food for thought, doesn't it? I am still upset I have not told all that to that student I argued with.

[1] The experiment is described in T. Bower's *Development in Infancy*, W. H. Freeman and Company, San Francisco, 1982 (2nd edition).

We all wish our kids to grow up bright and alert, don't we? What should we do to achieve it?

Urie Bronfenbrenner, in his book *Two Worlds of Childhood: US and USSR* tells about a project called "The Thirty-year Experiment." Its goal was to mainstream mentally retarded children living in a special home, to help them achieve independence. The experiment consisted of many stages, but probably the most heart-rending was the initial one. Each child was given a surrogate "mother"; these were mentally deficient women who lived in the same homes. Special tests carried out two years later showed that the children's intelligence increased 20–30% on the average, while the intelligence of the control group decreased. I was most impressed by the fact that these mothers were clearly unable to "teach" anything to these children. All they could do was to cuddle them, swaddle them, kiss them, and in general to "mother" them. And it happens that, at least at a certain age, emotional links and parental cuddles are much more important for the child's development, including here — and I would really like to emphasize this — their intellectual development than any other forms of teaching or activities. Parents should not forget about it!

I don't want to write a psychology book (especially since I'm not exactly the greatest expert). But I want to return to Piaget's phenomena one more time and describe an experiment, the only one that partially succeeded in teaching the law of conservation. I have in mind "cognitive conflicts" of Jan Smedslund (described on pp. 374–5, in the book of John Flavell mentioned above; the italics are mine.)

> If, for example, a given subject was inclined to think that elongating a plasticine ball augmented its quantity and that subtracting a piece from it diminished its quantity, the experimenter would do both at once .... Such a procedure was intended to give the subject pause, *to induce him to vacillate between conflicting strategies*; thence, he would be expected slowly to veer towards the simpler and more consistent addition-subtraction schema ...

Notice that nothing was "explained" to the children, nor verified by means of a balance. They succeeded to "teach" four kids out of thirteen, and there was no way to "unteach" them later.

I know I am inclined to jump to conclusions without solid bases or else to contradict myself (just a few pages ago I insisted that it was not our goal to teach a child the conservation law, and here I am trying to explain how we can do that). Never mind! What I want to underline as the basis of my teaching is the following: make them pause; make them hesitate between two mutually contradictory strategies. An opposing viewpoint is that intelligence is a capacity to rapidly solve brain teasers. Once more, at a risk of pretentiousness, I propose that our goal is to raise children to be *homo deliberans* ("reasoning man"). See later for concrete examples.

Why do we need theories?

I am holding in my hands a remarkable book with a boring title: *Mathematical Simulation in Ecology: Historical and Methodological Analysis*.[1] The subject of this book seemingly has nothing to do with our discussion. But it really crystallized my thinking about the importance of "theories."

Among other things, this book deals with the classical Lotka-Volterra equations. The initial idea is rather simple. There are, for example, populations of foxes and rabbits; as foxes eat rabbits, the population of rabbits diminishes, and foxes do not have enough food. Now it is the population of foxes that diminishes; hence the life of the rabbits becomes less dangerous and they multiply. The foxes have an abundance of food, their population increases, in turn diminishes that of rabbits, and so on. This model is rather easily interpreted in terms of differential equations. Luckily, these equations can be solved explicitly (which is rare in this subject) and yield nice cycles and oscillations if both populations are considered as functions of time.

The theory is ready; now we must test it empirically. In fact, empirical observations, e.g., measuring two populations (not necessarily of foxes and rabbits but of any two species, one of each eats the other) do not give satisfactory results. No surprise, considering how many extraneous factors interfere with the populations. Any attempt to isolate these factors and take them into account is too complicated. One can carry out the experiment in the laboratory, with all parameters controlled, using species (like yeast) which are much easier to deal with than animals. But even in these cases, there has been no thorough statistical treatment (the experiment was made in the 1930s before mathematical statistics was well developed), so it is difficult to come to definite conclusions. Once researchers even found oscillations in the populations of hares and lynx near Hudson Bay. The problem was that the cycles were going in the wrong direction, as if hares were predators and lynx were prey. The article was sarcastically entitled, "Do hares eat lynx?"

In a word, we have failed to experimentally prove this model. What should we do? Should we just abandon the model? Certain philosophers — science critics — are of this opinion. But the authors of the book are not philosophers, they are working researchers, and they come to the opposite conclusion. In the attempts to prove (refute, specify, develop, modify) the Lotka-Volterra model, numerous useful measurements were taken and valuable experience was accumulated. We cannot express it in simple equations, but nevertheless ecologists know a lot more today than at the end of the 1920s. They would not have known where to begin, had not they got this initial impetus. They would have remained on the level of general declarations like "in nature everything is interrelated."

[1]Authors are V. N. Tutubalin, Yu. M. Barabasheva, A. A. Grigoryan, G. N. Devyatkova, E. N. Uger.

What we must not forget is that an author of a theory invests so much in his creation that he starts to believe it as if it were Holy Gospel. Being a dilettante, I can juggle theories, even contradictory ones, can invent theories without (or almost without) foundations and refute them the next day. There are psychological models in which I have full confidence, like Piaget's phenomena. There are others which I think are rubbish, among them the so-called "theory of step-by-step formation of mental actions" once so popular in the USSR, or Piaget's theory of infant language acquisition.[1] But if we approach theories without dogmatism, they all are worth consideration since they all provide us with food for thought, and in my case, with ideas for inventing new puzzles for kids.

In the same ecology book, I read the following parable. A group of tourists visited islands on the coast of the White Sea. They've heard that somewhere in the vicinity there is an island with a lake of fresh water abundant with perch. It is possible they are already on this island. But how to find the lake? It's not easy to wander the marshy, hilly, Karelian forest. They need a *theory*! The water from the lake must find a way out: it may be an outgoing spring and there may be a path along the spring. They walk along the coast and indeed find a spring and a parallel path. So far so good! They go up the hill by the path along the spring. Soon, however, both the path and the spring disappear. "We climb up a steep hill from which we can see nothing but forest. For some time we wander blindly and all of a sudden find a path which brings us to a lake." What's more, it is full of magnificent perch! The moral of the story: we need theory not to correctly reflect reality but *to begin doing something*, and then we'll see. (Though, as the authors write elsewhere, the right theory is better than the wrong one.)

It's high time for me to stop babbling and begin doing something; time to return to our circle.

[1] See the remarkable book by Steven Pinker, *The Language Instinct: How the Mind Creates Language*, Harper Perennial, 1994.

Chapter 3

Children and $\binom{5}{2}$: The Story of One Problem

This chapter is based on my article "Children and $\binom{5}{2}$", published in *The Journal of Mathematical Behavior*, vol. 12, #2, 1993.

I suppose my readers have already noticed that in our sessions devoted, for instance, to probability theory, there are no definitions, no formulas, and not even arithmetic. I use the term "probability theory" for lack of a better one. Without these traditional mathematical ingredients, what are we really talking about?

Before answering this question, let's ask another one: where does probability theory come from? What is its source? Clearly, like many other sciences, like arithmetic itself, probability theory emerged from observations of certain real-world phenomena, namely, random, unpredictable phenomena. And it is exactly these kinds of observations — fundamental to the formation of science — which are worth making together with kids. Well, not all of them, of course, just the simplest ones. Besides, kids are making them on their own; e.g., when they play games with dice. What we can do is just make the probabilistic nature of their observations slightly more evident, as well as introduce them to the fact that a probabilistic world is also quite varied. We can show them, instead of a dice, an irregular polyhedron, and they will see that this way the game becomes "unjust": some scores happen more frequently than others. Or you can propose a game where they have to add the scores on two dice. Here too the kids will sooner or later notice that they get 7 more often than 2. With these kinds of activities the only limits are our imagination and the capacities of kids. If they have understood something, if something has stayed in they head, this is excellent. If not — no problem, then we just "played together."

To paraphrase the same idea: what we are interested in is not science[1] itself as a ready-made product of the previous generations, but rather the observations that precede it and that once gave an impetus to its emergence.

[1] *P. Z.'s Note:* The Russian word for "science" used here (наука) has a somewhat broader meaning than the English word, and includes mathematics, as well as other scholarly investigation.

This chapter looks at just one example. Chapter I was devoted to a single session; here I focus on a single problem. Just one problem, but much to think about!

A combinatorial puzzle

Our problem comes from combinatorics. At one time, this subject was studied in the 9th year of ten-year schools (i.e., for 16-year olds). Later it was considered too difficult (remember the frightening binomial theorem!) and removed from the curriculum. But all the difficulties lay in the fact that the high-school students start with the formulas without having the preceding "palpable" experience. I insist on the word "palpable" because a necessary preliminary stage would be manipulating real physical objects in order to count them; this is what combinatorics is about — counting various combinations of objects. However, in a theoretical presentation objects are lacking; you have to imagine them, as well as their combinations. It would have been much better if we could have started by counting real objects such as bricks or chips, but who would ever think of doing this in a high school?

We sit down around our pegboard. The assignment is to make chains of "beads" using five pegs, two of which will be red and the rest blue. This can be done in several ways. Our task is to sort out all the ways to do it and to avoid repetition.

Technically speaking, these sequences are called *combinations of two elements from a set of five*. In the Russian tradition the number of these combinations is designated by C_5^2. English-speaking countries use the notation $\binom{5}{2}$ instead. The value of this number is equal to $\frac{5\cdot 4}{2} = 10$.

Of course, the kids have no idea of all that and will not learn it at our sessions. They just make bead chains, in turns, one after another. We check each chain together: is it a new one or was it made earlier? Sometimes we disagree. For instance, does Figure 34 depict one or two different solutions? In fact, there is nothing to dispute: we can just *decide* that these solutions are either the same or different. This way we'll have two different problems, both worth solving. It turns out that it's easier to solve the problem where these solutions are considered to be different, so I propose this interpretation.

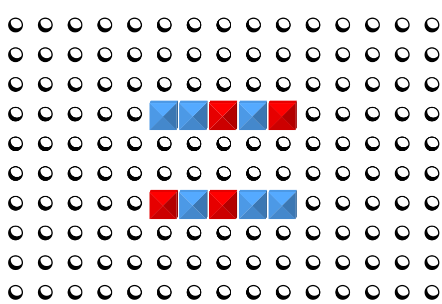

Figure 34. Are these chains of beads different?

Eventually we arrive at 10 solutions (distinct chains).

The main question in combinatorics is how many solutions there are. But the boys are nowhere near ready for this. At this point, they don't see the difference between "this is impossible" and "I cannot find any more" and are absolutely sure that I will be able to find the eleventh solution, the twelfth one, and as many as I wish. Consequently, I cannot leave them to their own devices. The kids worked haphazardly, without any system. In contrast, I am a model of organization: I sort out the solutions in a strict order. I put the first (leftmost) red peg in the first place and the second one in the second, third, fourth and fifth position in turn. When this sequence is exhausted, I put the first peg in the second position, etc. Do you think the kids are impressed? Not in the least! The only thing they comprehend is that I have also failed. (I could easily undermine my authority...) They can distinguish one solution from another, but are not yet able to distinguish order from disorder. We must postpone this problem and attack it again in, say, six months. (And it might be nice, meanwhile, to teach them to put their toys away where they belong. Perhaps order with toys is related to order in reasoning!)

Equivalent problems

After six months or so, the problem comes up again. Of course, I change its outward appearance. Each kid gets a sheet of paper on which there are drawn several rows of five connected circles (uncolored "beads") (Figure 35).

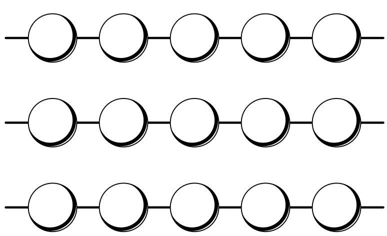

Figure 35. These beads are drawn on a sheet of paper. The children have to color two circles out of five so that the beads are different.

I have prepared more than a dozen such rows of "beads," in case of mistakes and repetitions. The task is to color two beads in each string, leaving the remaining three empty. He who finds the largest number of solutions will be the champion. And one more detail, insignificant at first sight. I hand out different colors of magic markers, but later, discussing the solutions, ignore this fact altogether: two circles can be colored by any color whatsoever. Children do not always understand which detail is important and which has nothing to do with the question and so I try, as best as I can, to emphasize purely combinatorial nature of the puzzle. I remember, with another group of children, instead of five circles, I drew five squares, another time five triangles, etc.

After a few minutes of independent work (and it was evident that it was harder to solve a puzzle on paper than with the pegboard, even though the kids were six months older), they noisily exchanged results and opinions. Now everybody has 10 solutions.

"Do you remember a similar puzzle we had once?"

How easy it is to be trapped by substituting my own point of view for the kids'! What does it mean — similar? I take it for granted that a similar puzzle concerns combinations of two elements out of five. The kids think, however, that a similar puzzle was when they drew something with magic markers. I hate prompting them, but this time I have to. The boys gladly grab the pegboard, make beads and even think of comparing the pegboard and paper solutions. Someone remembers that the previous time we also had 10 solutions. This gives rise to their first doubts:

"Perhaps, it's really impossible to make more?"

I smile an enigmatic smile and pass on to our next assignment.

I have discovered a gold mine. Or, it would be better to say, a new Proteus. This puzzle can be couched in an amazing variety of forms and can be revisited almost indefinitely. Here is another version. Each participant gets a sheet of graph paper marked with 3×4 rectangles. (Before I can get to the problem, we first argue briefly whether these regions are square or not.) The boys have to draw all possible paths from the lower-left corner to the upper-right one, under one condition: the moves can only be up or to the right (Figure 36).

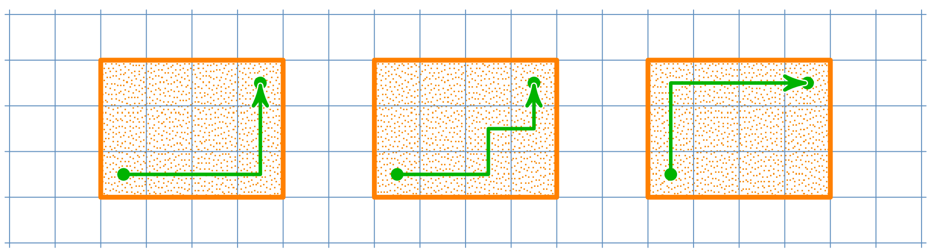

Figure 36. Find all possible paths from the left lower corner to the right upper one.

If you don't see the immediate connection between this puzzle and the previous one, be patient: you will understand it soon!

The work hums along. My "mathematicians" have obviously matured; they make fewer mistakes and find all 10 solutions quite rapidly. (I am afraid another trap awaits us: soon the boys will get used to the fact that the number of solutions in all combinatorial problems is 10. Once — not today — one of them called it "the problem about 10." I must take preventive measures and propose a puzzle with a different number of solutions.) At last I get to the main question: *How many steps upwards and how many steps to the right must be made in order to get from the left lower corner to the right upper corner?* Alas, it does not work. For me, a step is any passage from a cell to the adjacent one, but the kids think this is any straight segment.[1] First we need to agree on the definition of "step." This done, I am sure the answer will now be obvious. Nothing doing! I am bewildered and have to think it over after the session. Indeed, only my thoughtlessness could make me think that this question was easy. The coordinate representation of vectors, i.e., the fact that when we add vectors we also add their coordinates, is based exactly on this fundamental idea: that the number of horizontal and vertical steps is the same for all paths. I vividly remember being struck by this back in high school. This property of vectors can inspire an entire series of puzzles and could even help us to make a first approach to negative numbers, if reverse steps are allowed and denoted with the symbol of minus. (Unfortunately, I did not follow through with this.)

Meanwhile, we count steps: it turns out that each path contains exactly three steps to the right and two steps upwards. At the next session, we solve a "new" puzzle: we write sequences of letters UURRR, URURR, URRUR, etc. — each containing three letters R and two letters U (U for a step upwards and R for a step to the right; see Figure 37).

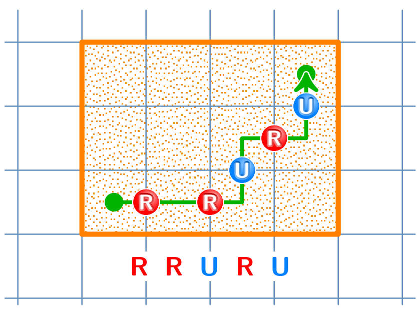

Figure 37. A move to the right is denoted by R, a move upwards by U.

You should have seen the excitement of the kids when I draw their attention to these symbols! They demand that the sheet of paper with five-letter sequences be cut into ribbons and start comparing them to the corresponding paths, pushing one another aside. I sit and watch them, then, as if by chance, try to introduce another idea.

"We can probably find more solutions, the eleventh, twelfth, . . . ?"

Gene answers,

"No, we have 10 here and 10 there".

"But perhaps they are different? Here we have 10 solutions, and there 10 other solutions?"

But by this moment all the paper ribbons are fixed and we see this is not the case: both groups of 10 solutions match beautifully. Or, in mathematical terms, they are in one-to-one correspondence. But it was worth doubting the result to better appreciate it afterwards!

With this enthusiasm, we can probably move a bit forward.

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[1] *P. Z.'s Note:* For example, in the second path of Figure 36, Zvonkin would count three right steps and two up steps — indeed, this will be true for all paths — but the kids would count two of each.

"Tell me, boys, can we denote our steps to the right and upwards with other letters? Not with U and R but with others?"

"Sure, any letter will do!"

"Which, for instance?"

"For instance, A and B", says Pete.

"Or hard sign and soft sign[1]," says Dima.

"Or," I say, "we can denote a step to the right by a plus sign and a step upwards by a comma."

"Ho-ho-ho!," they chuckle.

"Or," I go on impassively, "we can denote a step to the right by a white circle and a step upwards by a colored one."

"How?"

"This is how." I take the path that you can see in Figure 37, choose the corresponding five-letter string RRURU and draw nearby beads from Figure 38. In the ensuing pause — a pause before the explosion — I still have time to link the circles, which makes them look exactly like in the second version of my puzzle. They have recognized it! There is no doubt: the insight is accompanied by joyful howls and dancing. On the table everything becomes a mess and it is impossible to go on. It's time to end the session. Now I can step back, take a month off, and propose different types of puzzles. Let them get accustomed to this idea, let it take root. Besides, the kids may be sick of this type of puzzle.

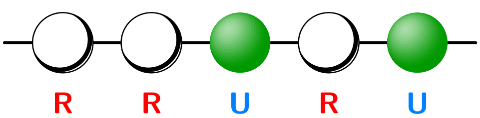

Figure 38. Draw a white circle instead of R and a colored circle instead of U.

Denoting...

We're approaching the finish line. On the table there are five empty matchboxes and two balls. We have to put the balls into two boxes, leaving three boxes empty. And we want to do this in different ways. (Figure 39).

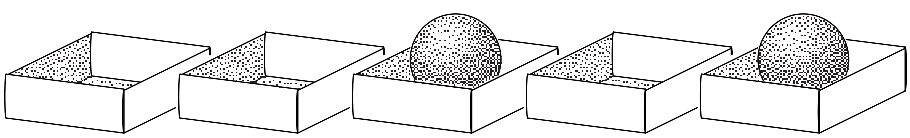

Figure 39. Put two balls into five boxes in different ways. The main difficulty now is to remember all the versions we have had, since they are no longer in view.

The work starts quite briskly but soon, after several trials, an animated discussion breaks out as to whether we repeated a solution. The boys ask me to be their judge but I pretend not to remember. This situation is, of course, intentional: I could have easily prepared enough matchboxes and

[1] These are two letters in the Cyrillic alphabet.

balls to provide several rows of five boxes; this would have led us to the same puzzle as before. Now, with a limited number of boxes, we have to compare each solution with those we have already found but which exist only in our heads. How to proceed?

Actually, not every kid will think of a way out. The point is that we shall need a symbol to *denote* each empty box and another symbol to *denote* a box containing a ball, and we shall have to note all the solutions using them. But behind this modest word *denote* is hidden an immense idea born and grown side-by-side with human civilization, namely, the largely mysterious history of writing: the evolution of pictograms into hieroglyphs, and these latter into alphabetic writing, and so on.

Mathematics has been engaged, from the very start, with developing systems of notation, first for numbers, then for arithmetic operations, then variables, and then for more and more abstract entities. As late as the twentieth century, the study of sign systems has occupied an independent discipline, semiotics.

In Chapter 5, we'll return to a more detailed discussion of this. Here I'll just remark that at our sessions, I've always tried not just to solve puzzles, but to formulate, at least for myself, more general goals. One such "meta-problem" is to introduce the ideas of semiotics. We often discussed the fact that numbers were denoted by figures, speech sounds by letters and musical sounds by notes. We also spoke about other semiotic systems, like road signs. So this idea is not entirely new for the boys. That's why they propose now to *draw* the solutions. At the beginning they try to make realistic sketches — apparently they are yet at the stage of pictograms. But this is not that easy technically, so we pass to the stage of hieroglyphs: the sketches become more abstract, an empty box is represented by a square and a box with a ball, by a square and an inscribed circle. I suggest that in the second case they simply draw a circle. Another difficulty lies in the fact that the kids are unable to draw accurately, so their squares and circles often look similar. I make another suggestion: to draw a crisscrossed circle. After implementing all these suggestions one of our solutions looks like Figure 40.

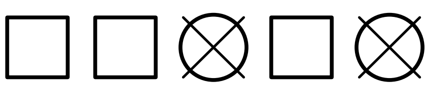

Figure 40. The first, second and fourth boxes are empty; the third and the fifth contain a ball.

"Why crisscrossed?"

"Why not? You can denote it the way you like," I shrug my shoulders to hint at the idea that a semiotician would formulate like "relative independence of the sign with respect to the signified and its (limited) arbitrariness."

It has happened that the resulting puzzle is more difficult that the previous ones in one aspect: each new solution is to be compared not with other solutions but with their representations. This time the boys find only nine solutions and after a few failures conclude there are no more.

At last comes the moment of triumph that I have been waiting for and preparing for so long. Pete exclaims, poking his finger at the sketch,

"Look! R, U, R, U, R!"

Dima, very excited, jumps up, "Yes, Dad, I was going to tell you!"

"Then there must be another solution," echoes Gene.

"Let's bring the solutions of that problem and see which one is missing," says Dima.

As always with kids, no sooner said than done: he is off to my study to fetch the list. But he does not need to go far. "Coincidentally," the envelope with the solutions turns out to be right here, on the table.

> *I was extremely disappointed that I "did not need to go far." First, I ran off to another room for nothing; second, I understood there was no real discovery and that Dad had prepared everything beforehand. — Dima.*

We discuss which version — with letters, with paths or with beads — is more convenient for our purposes, and choose beads. While we lay out stripes of paper with "beads" on them, there is a little confusion: one stripe was accidentally rotated by 180°. Consequently, one of our previous solutions is lost and another one, symmetric to it, is represented twice. This almost derails us.

> *I wanted to say the stripe was turned but didn't, as I thought it was probably not important. — Dima.*

Somehow all the boys are sure that the missing solution will be the last one. But the fact we find it as the fourth solution does not discourage them in the least. They place balls representing this solution, tell me how to draw the 10th one and then match the remaining beads with other sketches. I finish the session feeling absolutely triumphant.

Today's events were really important. We didn't just solve a puzzle. We reduced it to another, isomorphic one, which had already been solved. This is an extremely important mathematical idea, and isn't it miraculous that it was possible to demonstrate it to six-year-old kids? And on top of this, they were able to discover it on their own!

Proof

Our circle zooms along with dizzying speed. Hardly have we sorted out one great idea, when another, no less great, is knocking at the door. Why do we always have ten solutions? The question seems to arise quite naturally. Is it true there is no more? Or have we just failed to find more? How to prove there are only ten of them?

In other words, we are approaching the idea of *proof*: The central — in fact, formative — notion of mathematics which sets it apart from all other intellectual disciplines. The concept of what is and what is not a proof had been evolving throughout centuries and became what it is only at the turn of the twentieth century. Mathematicians of preceding generations would have accepted as convincing the statements that would nowadays be indignantly rejected by any school teacher. Actually, we are dealing here with a very odd phenomenon. Why would abstract reasoning, often completely beyond any common sense, make any statement more convincing to us? A very intelligent high school student once asked his teacher,

"It's perfectly obvious that the angles at the base of an isosceles triangle are equal; you can see it in examples. Nevertheless, this fact has to be proved. On the other hand, it is not at all obvious that voltage is equal to current multiplied by the resistance. Yet this fact is never proved; it is only demonstrated by experimentation. Why?"

Such intelligent questions are rare. Most often than not, pupils perceive proof as a ritual: "This is how you are supposed to act in mathematics." This brings to mind a historical anecdote from the eighteenth century. A nobleman dabbling in mathematics says to his teacher, "Who needs all this ambiguous reasoning? We are both noblemen, give me your word of honor that the theorem is true, and it will be enough for me."

But doesn't the same occur when we read a textbook of history? No proof, just "theorem formulations:" it happened there and then. Period. And the "word of honor of a nobleman" — in this case, the textbook author — is enough for us to believe it. As a matter of fact, a mathematician's everyday activities do not radically differ from those of a historian. It would be an illusion to believe that a mathematician finds a proof and stops at that, because in the overwhelming majority of cases he produces incorrect proofs. But he is aware of the fact that the same method will do to prove another, obviously false statement and so he perseveres, looking for errors and contradictions, and trying to discover a new road to the truth. He *is discovering new territory.* And he will stop only when all the parts of the puzzle fit. A historian or any other researcher is also looking for the same kind of harmony. But in the textbook we will merely see the shortest way from A to B. The moment the student swings from this shortest way, "turns to the right one traffic light too early," he finds himself in a completely unknown place and has no idea how to get out. Not so with a real expert: he has perfect mastery not only of the shortest way but also of the entire neighborhood, since he has been studying it thoroughly for a long time.

But let's not wander too far afield; we must get back to our kids. There's a paucity of material for introducing the idea of proof to little ones, but there are some possibilities. For example, "odd one out" puzzles with different possible solutions, where it is important not only to give the correct answer but also to explain it. Or puzzles like this: prove that we see with our eyes and hear with our ears, and not vice versa (proof: close your eyes and you can't see; plug your ears and you can't hear); prove that clouds are nearer to the earth than the sun (proof: clouds cover the sun); prove that we think with our head and not with our stomach. I personally do not know a convincing proof for the last puzzle[1] but at our session I proposed the following: cut off a man's head and he will stop thinking. The boys did not agree but none said this proof would also be valid for the stomach.

Well, what could be a proof for our combinatorial puzzle? Clearly, it must be an *ordered* enumeration of possibilities, i.e., a kind of enumeration that will make us certain no version has been lost. A year ago the boys were unable to understand this idea. Have they become more mature by now?

Let's return to the discussion cut short in the middle of a sentence. So, how can we be sure that there are no other solutions besides those 10 already found?

Dima says, "We must try for many years and if we don't find any, then there is none."

I argue, "And if still there is one?"

Gene is pessimistic, "I won't be able to find more."

Pete asks me if it is true that I don't know the number of solutions or I do know but ask them just for the sake of talking. I confess that I know exactly how many solutions there are. Now the boys cease to understand what is it that I want. At this moment I am saved by Dima who utters vague phrase — I don't really catch its meaning as I am thinking about something else, but I focus on the words "the first box from the left." I hasten to interpret these words in the necessary direction. This is what we are going to do: take the first ball and put it in the first box from the left. Where we can now put the second ball? Obviously in one of the remaining boxes: the second, the third, the fourth or the fifth one. Thus we obtain 4 solutions. Having exhausted all the solutions with the first ball being in the first box, let's now put it into the second box. Again we'll have 4 positions for the second ball: we can put it into any empty box. Now put the first ball into the third box, and so on, and so forth. In other words, we'll get 4 solutions 5 times, that is... 20 solutions! What do you think of that? The boys are utterly bewildered, and I end our session as quickly as I can.

This time I am sure I've hit the bullseye! They will certainly puzzle over it, trying to understand why, to get the correct answer, that 20 must be divided by 2.

[Damn it! What a shame. My excruciatingly honest son (of today) compels me to admit that I've stretched the truth a bit in this story. It was just wishful thinking. At the time, I was terribly upset that I had not had this idea during the session! In reality I did explain to the boys why there would be 10 and not 20 solutions and why, having put the first ball into the second box, we did not have to put the second ball into the first box. But what a pity to waste such a beautiful story!

After the session we had a long argument with Dima. I didn't take notes, so I don't remember what we were talking about but Dima does.

[1] Apparently no such proof exists. Quite recently I learned that ancient Egyptians, when making a mummy, neatly preserved for future life all the internal organs except the brain which they threw away as completely useless.

> *At first Dad's systematic enumeration did not convince me. Either at the end of the session or after it I asked him, "But what if you still missed one?"*
>
> *"Which one?"*
>
> *"I don't know. One of them."*
>
> *"Well, in which box is the first ball?"*
>
> *"I don't know. In any box."*
>
> *"Let's say in the second."*
>
> *"OK, in the second."*
>
> *"But then the second ball can be either in the 3d, in the 4th or in the 5th box, and we have already enumerated them."*
>
> *At this moment I felt (though perhaps I was not ready to recognize it) that indeed the 11th version would be hard to squeeze through: no matter where we put the first ball, it will always turn out that we have already enumerated all the possibilities. — Dima.*

Perhaps after all there was no reason to be upset. My reasoning, even as such, was hard to understand, and I would have made it harder by piling one difficulty onto another.]

Physics and logic

I would like to relate another one of our discussions with Dima which occurred when he was 5 years and 9 months old. Looking back, I find it odd that I could discuss such serious subjects with such a little boy. Nevertheless, it is documented in my journal. (For some time already I have been entering here not only our sessions but other related stories.)

The discussion starts with a sudden and somewhat unusual question: Does God exist? As a rule I try to avoid a direct answer to this question (besides, I don't know the answer), thinking that he will decide this for himself when he grows up. This strategy has not been successful, since I am not his sole companion: someone has already told him that "there is no God because nobody has seen him." As usual, I steer the discussion to suit my goals and say that if this is the case, how can he convince me that a *dream* exists: nobody has seen it, has he?

Dima tries several tentative approaches,

"What is your dream?"

I answer I don't have any.

"But what do you wish more than anything?"

I say that I wish nothing.

"But you do wish me to grow up intelligent?"

This is a sort of moral pressure but I remain adamant: I say I do not wish anything at all, period.

Dima falls silent and thinks; then he asks the question that made me want to tell this story: He asked me how it is possible, at all, to convince another person of something?

"There are different ways to do that," I say. "For instance, in mathematics, we use *proofs*."

"What do you mean?" asks Dima.

I remind him of our recent puzzle with 10 solutions and of our attempts to systematically enumerate them, to be sure we hadn't missed any.

"And in physics, we do experiments."

"Ah, I get it." (By this time we had already looked at L. L. Sikoruk's *Physics for Kids* and done some of the experiments in it.)

"For instance, what do you think: which objects fall down faster, heavy ones or light ones?"

"Of course heavy ones."

"That's what you think; and someone else will say that all objects fall with the same speed."

"No, no, this is not true!"

"Why not?"

"But if we take a stone and a sheet of paper, the stone will fall faster."

"So, to convince this person you will have to make an experiment, won't you? You will take a stone and a sheet of paper and will see which one will fall faster."

"Yes."

"Now let's do another experiment."

I got the idea for this from a friend. First we take two sheets of paper and they surely fall down with the same speed. Then I crumple one sheet of paper into a ball. I am going to ask Dima which sheet will fall faster but he stops me,

"Now this one (he indicated the paper ball) has become heavier."

"Why!?!"

"Because it will fall down faster."

So this is how things are. For a physical experiment to convince you, your logic must be mature enough to be aware of the inadmissibility of circular logic. No conclusions can be drawn from experiments without logic. Which one is primary and which one is secondary? Frankly, I have no idea. I suppose they grow up together in a kind of symbiosis.

There is no stopping me now. We go on dropping all kinds of objects we can lay our hands on: a button and a heavy sheet of drawing paper; a button and a weight; a hollow plastic cube and a wooden cube of the same size, etc. Dima is clearly bewildered by the results. He even ventures a hypothesis that a button is heavier than a sheet of drawing paper, but then gives it up in view of its obvious absurdity.

"Well, it means sometimes light things fall down faster and sometimes heavy things do."

He is almost ready to be satisfied with this quasi-theory but then — eureka!

"I got it, Dad! That's the air that does not let it fall!"

"Does not let who?"

"The sheet of paper is big and the air does not let it fall down, and a button is small, and the air stops it less."

"Good! And if there were no air, what would happen?"

"Then they all would fall down with the same speed!"

"Right! And what happened when I crumpled the sheet of paper?"

Dima chooses his words to give me a correct answer but I am too impatient and answer in his place,

"The air does not stop it."

Dima corrects me, "Yes, it still does, but it's going to stop it less."

I have already shared with the reader one of my guiding principles: never impose your own point of view on a child, even by a hint. But the hierarchy of principles contains another principle, even more important: never follow your principles blindly. Perhaps now is a good time for flexibility. And with an obvious hint at the "only correct answer" I indicate again the crumpled sheet of paper and ask,

"Does it really become heavier?"

Dima laughs a knowing laugh, as if wishing to say, "It's hard to believe I could say such a stupid thing!" and says,

"Of course, not! Maybe only a tiny bit heavier."

In the evening, when I am noting this conversation into the journal and thinking it over again, I remark a thing that has escaped me before. What we have done together is not a physical experiment properly speaking. An experiment is a question we ask nature, *an answer to which we don't know.* In our case Dima knew all the answers beforehand. Strictly speaking, it was not necessary to really throw a weight or a button — the child's personal experience of the surrounding physical world is enough to predict correctly the result of this experiment. We may say that none of these experiments has given him new information — if we limit ourselves to facts. What was new was comparing and arranging known facts. As a matter of fact, we have made the same systematic enumeration of logical possibilities that we had earlier performed for the balls in the matchboxes. This situation shows once more why questions are so important in teaching. They help a child to compare the experiences that once existed separately, on different shelves of memory's bookcase.

. . .

During the summer, we rented a country house not far from Moscow, and one day Pete came to visit us. The boys talked about their recent visit to the zoo where they were shown monkeys. I interrupted, telling them that they were not shown monkeys, they were shown to monkeys. This naturally provoked a heated protest, but they did not immediately find the right argument.

"We were looking at them."

This was easily countered with, "You were looking at them — big deal! They were also looking at you."

Their second argument was more solid: "We can walk wherever we want to, and the monkeys can't."

But I still refuted them with, "No, you don't walk wherever you want to. You can't go inside the cage. And the monkeys can't go outside the cage. There are bars separating you, and you walk wherever you want to on one side of them, and the monkeys do the same on the other side."

We were arguing this way for some time until Dima exclaimed with delight, as if he had caught me red-handed,

"Hey, Dad! We're doing math again!"

Quite an interesting evolution: at the first session of our circle the kids rushed to count buttons laid out on the table. That was how they saw mathematics then. Now it has become a sort of logical game, Lewis Carroll-style.

. . .

It's a shame that I've robbed the following chapter of its tastiest morsels. But first, I wanted to present the material with a new perspective. I was not sure that the reader could have followed the evolution of the combinatorial puzzle if it had been divided amongst several chapters. Second, the events at the beginning of the chapter belong to the "undocumented" period and would have otherwise remained off-screen. I hope there are still items of interest left in the chapters to come.