2000 Olympiad

Level A

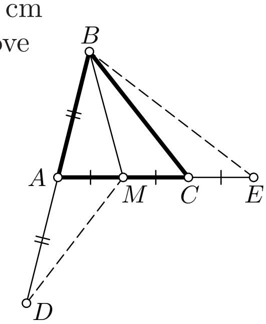
Problem 1. Two different numbers x and y (not necessarily integers) satisfy $x^2 - 2000x = y^2 - 2000y$. Find the sum of x and y.

Problem 2. Twelve parties took part in a parliamentary election. There were 100 seats to be filled, and to win a seat, a party must obtain *more* than 5% of the votes. The parties that win seats distribute them among themselves in proportion to the number of votes received: if party A got x times more votes than party B, it will also get x times more seats than party B.

In this election, every voter chose exactly one party: there were no invalid ballots, write-in votes and the like. Each party won an integer number of seats. The Party of Math Lovers got one quarter of the votes. What is the largest number of seats this party can gain? (Explain your answer.)

Problem 3. The bases of a trapezoid are m cm and n cm long, where m and n are distinct positive integers. Prove that the trapezoid can be cut into congruent triangles.

Problem 4. The median BM and the side AC of a triangle ABC have equal length. Points D and E are taken on the extensions of the sides BA and AC, respectively, so that $AD = AB$ and $CE = CM$ (see figure). Prove that the lines DM and BE are perpendicular.



Problem 5. Some of the cards in a stacked deck are face down and the rest are face up. Now and again Pete draws from the deck one or more contiguous cards, subject to the condition that the first and last of them are both face down. He turns over this set of cards as a unit, and inserts them back into the deck in the same place. Prove that sooner or later all the cards in the deck will be face up, no matter what Pete does.

Problem 6. What is the greatest number of chess knights that can be placed on a 5×5 chessboard so that each knight attacks exactly two others?

Give an example and explain why an arrangement with more knights is impossible. (A knight in position (a, b) attacks the eight squares in positions $(a \pm 1, b + 2)$, $(a \pm 1, b - 2)$, $(a \pm 2, b + 1)$, and $(a \pm 2, b - 1)$—or as many of them as fall within the board.)

Level B

Problem 1. Solve the equation

$$(x+1)^{63} + (x+1)^{62}(x-1) + (x+1)^{61}(x-1)^2 + \cdots + (x-1)^{63} = 0.$$

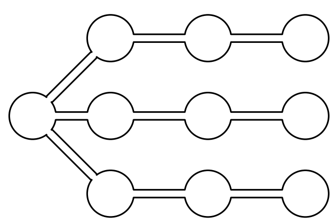
Problem 2*. Twenty-three positive integers, not necessarily distinct, are written in a row. Prove that we can insert parentheses and plus and times signs between them so that the value of the expression thus obtained is divisible by 2000.

Problem 3. Let A be a point inside a given circle. Find the locus of the vertices C of all rectangles $ABCD$ whose vertices B and D lie on the circle.

Problem 4. Greg filled the squares of a chessboard with the numbers 1, 2, 3, …, 63, 64 in some order and is willing to tell Linda, for each rectangle made up of two squares, the sum of the numbers in them. He adds that the numbers 1 and 64 lie on the same diagonal. Prove that this information is enough for Linda to determine exactly the numbers in every square.

Problem 5*. The circles having as diameters the sides AB and CD of a convex quadrilateral $ABCD$ are externally tangent to each other, at a point M distinct from the intersection of the quadrilateral's diagonals. Denote by K the second intersection of the circumcircle of triangle AMC with the line defined by M and the midpoint of AB, and denote by L the second intersection of the circumcircle of triangle BMD with the same line. Prove that $|MK - ML| = |AB - CD|$.

Problem 6*. The stage of a video game is made up of round shelters connected by tunnels; a possible configuration is shown in the illustration. Your target is in one of the shelters, but you cannot see it. You can blast one shelter at a time, and if it's the one where your target is, you win. In between shots the target *must* cross a single tunnel into a neighboring shelter; it doesn't matter if that shelter has been hit before.



You have a winning strategy if you can plan a sequence of shots that will eventually hit the target no matter where it starts and what moves it makes.

(a) Prove that the configuration in the figure does not admit a winning strategy.

(b) Find all configurations that do not admit a winning strategy, yet acquire one as soon as any of their tunnels is blocked.

Level C

Problem 1. Two points, A and B, are marked on the graph of the function $y = 1/x$, $x > 0$. Denote by H_A and H_B the feet of the perpendiculars dropped from these points to the x axis and by O the origin. Prove that the area of the figure bounded by the lines OA and OB and the arc AB of the graph equals the area of the figure bounded by the lines AH_A and BH_B, the x axis, and the arc AB.

Problem 2. Let $f(x) = x^2 + 12x + 30$. Solve the equation

$$f(f(f(f(f(x))))) = 0.$$

Problem 3. A convex polygon is drawn on graph paper so that all its vertices are on grid intersections and none of its sides is horizontal or vertical. Consider the segments of vertical grid lines formed by intersection with the (filled) polygon. Show that the sum of their lengths is equal to the similar sum for horizontal grid lines.

Problem 4*. See Problem 5 of Level B.

Problem 5. A sequence $x_1, x_2, \ldots, x_n, \ldots$ will be denoted by $\{x_n\}$. Given two (possibly identical) sequences $\{b_n\}$ and $\{c_n\}$, you can form the sequences $\{b_n{+}c_n\}$, $\{b_n{-}c_n\}$, $\{b_n\, c_n\}$, and $\{b_n/c_n\}$ (if all c_n are distinct from 0). Also, from any given sequence, a new sequence can be formed by deleting finitely many initial terms.

(a) Starting from the single sequence $\{a_n\}$, where $a_n = n^2$, can these operations lead to the sequence $\{n\} = 1,\ 2,\ 3,\ 4,\ \ldots$?

(b) What if $a_n = n + \sqrt{2}$ instead?

(c) What if $a_n = \dfrac{n^{2000} + 1}{n}$?

Problem 6. Seven cards were drawn from a deck, shown to everybody, and shuffled. Then Greg and Linda were given three cards each, and the remaining card was either (a) hidden or (b) given to Pat.

Greg and Linda take turns announcing information about their cards. Are they able to ultimately reveal their cards to each other in such a way that Pat cannot deduce the location of any card he doesn't see? (No special code was set up in advance; all announcements are in "plain text".)

Level D

Problem 1. If two numbers m and n are relatively prime, what is the highest possible value of the greatest common divisor of $m + 2000n$ and $n + 2000m$?

Problem 2. Compute

$$\int_0^\pi \bigl(|\sin 1999x| - |\sin 2000x|\bigr)\, dx.$$

Problem 3. The chords AC and BD of a circle with center O meet at a point K. Let M and N be the circumcenters of the triangles AKB and CKD, respectively. Prove that $OM = KN$.

Problem 4. Denzel has three sticks. If it is impossible to make a triangle with these sticks, Denzel shortens the longest of them by a length equal to the sum of lengths of two other sticks. If the stick did not disappear after this operation, and it is still impossible to make a triangle, Denzel repeats the operation, and so on. Can this process continue endlessly?

Problem 5. Each participant of a round-robin chess tournament plays one game against each other. A win is worth one point, a draw half a point, and a loss zero. A game is called *anomalous* if its winner ends the tournament with a score less than the game loser's score.

(a) Can anomalous games amount to more than 75% of total number of games in the tournament?

(b)* Can they amount to more than 70%?

Problem 6. Is it possible to arrange infinitely many congruent convex polyhedra in a layer bounded by two parallel planes so that no polyhedron can be removed from the layer without moving the remaining ones?

2000 Olympiad

Level A

1. Move $2000x$ to the right-hand side and y^2 to the left-hand side.
3. Extend the sides of the trapezoid until they meet.
4. Consider the triangle KBE, where K is point symmetric to M with respect to A.
5. Use a monovariant (see Fact 2).
6. The number of knights on white squares is equal to the number of knights on black squares.

Level B

1. Use the formula for $a^n - b^n$.
2. Suppose that we have seven numbers: four numbers divisible by two and three numbers divisible by five; then their product is divisible by 2000.
3. Apply the Pythagorean Theorem several times.
4. Linda can find out the difference between the numbers in any two squares of the same color.
5. Let O_1 and O_2 be the circumcenters of the triangles AMC and BMD. Consider the orthogonal projections of O_1, O_2, and the midpoint of O_1O_2 on the line mentioned in the problem's statement.
6. Consider the shelters where the target can be after an even number of shots.

Level C

1. The areas of the triangles OAH_A and OBH_B are equal to $\frac{1}{2}$.
2. Complete the square.
5. (b) All sequences that can be obtained from $\{n + \sqrt{2}\}$ are of the form

$$\left\{ \frac{P(n+\sqrt{2})}{Q(n+\sqrt{2})} \right\},$$

where P and Q are polynomials with integer coefficients.
(c) Consider the mapping that takes the sequence $\{a_n\}$ into $\{a_{n+1} - a_n\}$. Apply it to the initial sequence enough times.

6. (a) First, try to understand how Greg can make his cards known to Linda without letting Pat know anything about them.
(b) Let Greg and Linda number their cards from 0 to 6.

Level D

1. $\gcd(dm, dn) = d$, for any d.
3. Either the points O, M, K, and N are on the same line or the quadrilateral $OMKN$ is a parallelogram.
4. Choose the sticks so that after each truncation the ratios of their lengths remain the same.
5. (a) Let the number of participants be $2M$. Call *strong* the participants that took the first M places and *weak* all the rest. Consider three types of games: strong against strong, strong against weak, and weak against weak.
(b) Construct the table of a tournament in which all players have the same final score, and the number of ones above the main diagonal is approximately equal to a quarter of the total number of games.

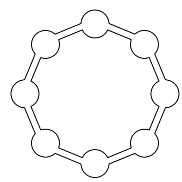
2000 Olympiad

Level A

1. $x + y = 2000$. **2.** 50 seats. **6.** 16 knights.

Level B

1. $x = 0$. **3.** If O and R are the center and radius of the given circle, the desired locus is the circle centered at O with radius $\sqrt{2R^2 - OA^2}$. **6.** (b) The configuration of part (a) and all configurations consisting of exactly one cycle of shelters, such as the one in the figure.



Level C

2. $x = -6 \pm \sqrt[32]{6}$. **5.** (a) yes; (b) no; (c) yes. **6.** (a) yes; (b) yes.

Level D

1. $2000^2 - 1$. **2.** 0. **4.** Yes. **5.** (a) no; (b) yes. **6.** Yes.

2000 Olympiad

Level A

Problem 1. Move $2000x$ to the right-hand side and y^2 to the left-hand side, to obtain

$$x^2 - y^2 = 2000x - 2000y.$$

Factor the difference of squares in the left-hand side:

$$(x - y)(x + y) = 2000(x - y).$$

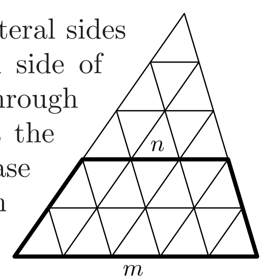
Since $x \neq y$, we can divide both sides by $x - y$ to obtain $x + y = 2000$.

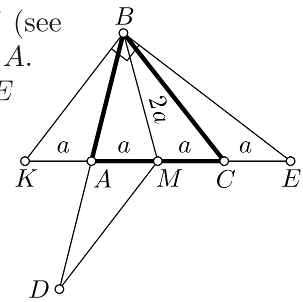
Problem 2. The idea of the solution is that the Party of Math Lovers (PML) gains the greatest number of seats if the total number of votes going to parties that don't get seats (because they got 5% or less of the total vote) is as large as possible.

If 10 parties each obtain exactly 5% of votes and two parties, including PML, 25% each, then only two parties will win seats in the parliament. Each will have exactly 50 seats.

Let us prove that PML cannot win a greater number of seats. If 11 parties fail to pass the 5% threshold, together they've received at most 55% of votes; but $55\% + 25\% < 100\%$. Therefore, at most 10 parties failed to meet the threshold, and they've received at most 50% of the votes. Hence, the parties that did get seats got at least 50% of votes. Thus PML's 25% of the total vote represents at most half the total vote, hence at most half the seats, or 50 seats.

Problem 3. Suppose that $m > n$. We extend the lateral sides of the trapezoid to form a triangle and divide each side of this triangle into m equal parts. The lines drawn through the division points parallel to the triangle's sides cut the triangle into congruent small triangles. The smaller base of the trapezoid is one of these lines (because its length in centimeters is integer), so the trapezoid is cut into congruent triangles.





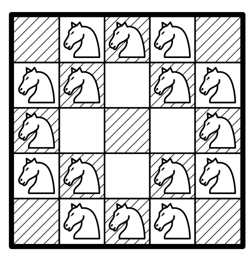
Problem 4. Let a be the distance from C to M (see figure). Let K be a point symmetric to M about A. Then $KM = MB = ME = 2a$. Therefore, KBE is a right triangle (see Fact 14), with $KB \perp BE$. Also, the quadrilateral $DKBM$ is a parallelogram, since its diagonals bisect each other. Hence $DM \parallel KB$ and $DM \perp BE$.

Problem 5. We will encode the arrangement of the cards in the deck by a number with as many digits as there are cards. The k-th digit of this number from the right is 1 if the k-th card from the bottom is turned its back up; otherwise, the k-th digit is 2. For instance, if all cards in the deck are turned down, the resulting number is 2222…222.

It is easily seen that after each of Pete's operations the code of the deck decreases. Indeed, let us compare the codes after and before such an operation. Consider all the digits that changed and take the leftmost, that is, the highest-order, digit. Obviously, this digit changed from 2 to 1. But this means that the encoding number decreased.

Since there are only finitely many n-digit numbers made of 1s and 2s, our number cannot decrease forever: eventually we will reach the number 1111…111, corresponding to the arrangement in which all the cards are turned up. See also Facts 2 and 11.

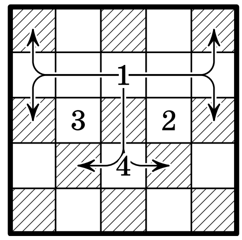
Problem 6. An arrangement of 16 knights satisfying the condition in the problem is shown in the figure. Let us show that a greater number of knights is impossible. Color the squares of the chessboard black and white as shown in the figure. We first show that the number of knights on black squares must be equal to the number of knights on white squares. Join all pairs of knights that attack each other by a line segment; then each segment joins a white square with a black one. Each square is joined to other squares by two segments. It follows that the number of segments equals twice the number of knights in white squares and, at the same time, twice the number of knights in black squares. Therefore, these two numbers coincide.



There are 12 white and 13 black squares. If the number of empty white squares is n, the number of empty black squares is $n + 1$, and it suffices to prove that $n \geq 4$, because in this case at least $4 + (4 + 1) = 9$ squares are empty.

For any optimal arrangement of knights, the central square is empty. Otherwise, six of the eight white squares attacked by the knight in the central square are empty, so $n \geq 6$, so the number of knights is at most $25 - 6 - (6 + 1) = 12$.

Consider the square marked 1 in the figure. If it is not empty, four of the six black squares attacked by the knight in this square are empty. Taking into account the empty central square, we see that in this case at least 5 black squares are empty: $n + 1 \geq 5$. Thus, we can assume that square 1 is empty. The same argument works for the squares marked 2, 3, and 4: they must also be empty. But then $n \geq 4$, as we wished to prove.



Level B

Problem 1. We multiply both sides of the equation by $(x+1)-(x-1) = 2$, to obtain after simplification

$$(x + 1)^{64} - (x - 1)^{64} = 0.$$

This follows from the formula

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

applied to $a = x + 1$, $b = x - 1$, $n = 64$.

This equation is readily solved: the condition $(x + 1)^{64} = (x - 1)^{64}$ is equivalent to $|x+1| = |x-1|$. Since $x+1 \neq x-1$, we obtain $x+1 = -(x-1)$, whence $x = 0$.

Problem 2. Divide the 23 given numbers into seven groups of consecutive numbers: three groups of five numbers each and four of two numbers each (the order of the groups is irrelevant). Each group is enclosed in parentheses, and multiplication signs are placed between the groups. If the signs inside the groups are inserted in such a way that the value of each two-number parentheses is divisible by 2 and that of each five-number parentheses is divisible by 5, then the entire expression is divisible by $2^4 \cdot 5^3 = 2000$.

We show that the desired arrangement of operation signs is possible. First, consider a group of two numbers. If these numbers are of different parity, we insert the multiplication sign between them; if they are of the same parity, we insert the plus sign. The result is obviously divisible by 2 (see Fact 23).

Now consider a group of five numbers a_1, a_2, a_3, a_4, and a_5 in this order. Write down the remainders of the following five sums when divided by 5:

$$a_1, \quad a_1 + a_2, \quad a_1 + a_2 + a_3, \quad a_1 + a_2 + a_3 + a_4, \quad a_1 + a_2 + a_3 + a_4 + a_5.$$

If any of these remainders is 0, the corresponding sum is divisible by 5. In that case, we insert pluses between the terms of this sum, put the sum in parentheses if necessary, and fill the remaining spaces between the numbers of the group with multiplication signs.

If none of the remainders is zero, then, by the pigeonhole principle (see Fact 1), at least two of them will be equal, since there are five sums and only four possible nonzero remainders. Suppose that the sums $a_1 + \cdots + a_i$

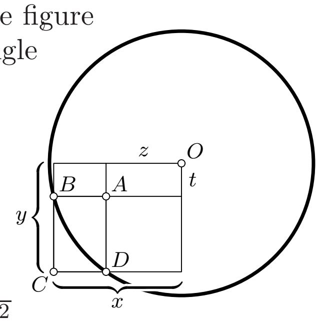
and $a_1 + \cdots + a_j$ $(i < j)$ yield the same remainder upon division by 5. Then the difference between these two sums is divisible by 5, and it equals $a_{i+1} + \cdots + a_j$. We form this last sum by inserting pluses between the numbers in it, place parentheses around it, and fill the remaining spaces with multiplication signs. Two examples follow:

$$(1+2+3+4)\cdot 3 = 30; \quad 2\cdot(1+4)\cdot 7\cdot 1 = 70.$$

Thus, we can always insert signs in a group of five numbers so that the result be divisible by 5.

Remark. This problem is a variation on the following classical piece: *in any string of n integers, one can choose one number or several numbers in a row whose sum is divisible by n.*

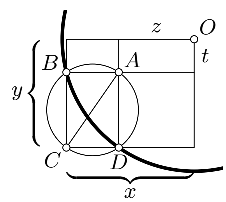
Problem 3. Let O be the center of the circle described in the statement of the problem. We prove that the point C satisfies $OC^2 = 2R^2 - OA^2$, where R is the radius of the circle. In the notation of the figure the segment OC is the hypotenuse of a right triangle with legs x and y. By the Pythagorean Theorem,



$$OC^2 = x^2+y^2 = (x^2+t^2)+(y^2+z^2)-(z^2+t^2).$$

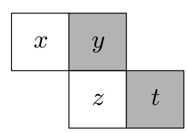
It remains to notice that $x^2+t^2 = y^2+z^2 = R^2$ and $z^2+t^2 = OA^2$ (again by Pythagoras). It follows that the distance from C to A is constant and equal to $\sqrt{2R^2 - OA^2}$, i.e., the locus in question is the circle with center O and radius $\sqrt{2R^2 - OA^2}$.

Conversely, let a point C satisfy $OC^2 = 2R^2 - OA^2$. We'll prove that there is a rectangle $ABCD$ with vertices B and D on the initial circle. To this end we construct a circle with diameter AC (see figure). It intersects the initial circle at two points. Denote one of them by B. Then $\angle ABC = 90°$ (see Fact 14). Let us complete the triangle ABC to a rectangle $ABCD$. It remains to prove that D is on the circle. In the figure's notation:



$$OD^2 = y^2 + z^2 = (x^2+y^2) + (z^2+t^2) - (x^2+t^2)$$
$$= (2R^2 - OA^2) + OA^2 - R^2 = R^2.$$

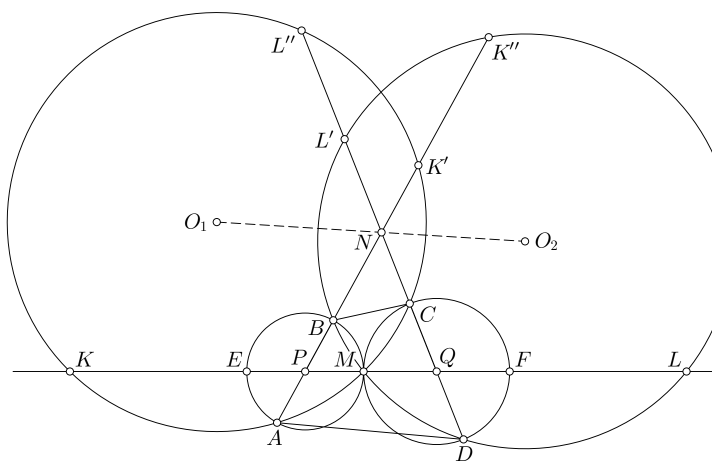
Problem 4. Suppose that Linda has lots of 1×2 dominoes. If she places two dominoes on the board so that they have exactly one common square — say covering x-y and y-z in the figure — then, by subtracting the sum of the numbers under the y-z domino, namely $y+z$, from the sum under the x-y domino, namely $x+y$, she will find the difference $x-z$ of the numbers in the two squares currently covered by a single domino (which have the same color).



The next domino will be placed so as to join a single-domino square with an as yet uncovered square — say, the domino zt in the figure at the bottom of the previous page. Adding the difference obtained above with the sum of numbers under the new domino, Linda will obtain the sum $x+t$ of the two numbers in squares currently covered by a single domino (which now have opposite colors). Proceeding in the same way, Linda can add new dominoes one after another to construct a chain joining any two squares. If these squares are the same color, she gets to know their difference; otherwise, their sum.

It is known that 1 and 64 are on the same diagonal, and so on squares of the same color. Their difference is 63, whereas the difference of any two other integers between 1 and 64 is less than 63. Therefore, Linda can determine which two squares contain 1 and 64. Now, knowing the sum of (or the difference between) 64 and the number in any other square, she can determine the positions of all numbers.

Problem 5. *First solution.* Let P and Q be the midpoints of AB and CD, and let O_1 and O_2 be the centers of the circles drawn through A, M, and C and B, M, and D, respectively. Denote by H_1 and H_2 the projections of O_1 and O_2 on the line PQ:



Step 1. The points M, P, and Q are collinear. Indeed, the lines PM and QM contain the radii of the circles tangent at M, and hence they are perpendicular to the common inner tangent of these circles.

Step 2. The points P and Q lie on the circle with diameter O_1O_2. Indeed, PO_1 and PO_2 are perpendicular, being the perpendicular bisectors to the segments MA and MB, because M lies on the circle with diameter AB (see Fact 14). Similarly, $QO_1 \perp QO_2$.

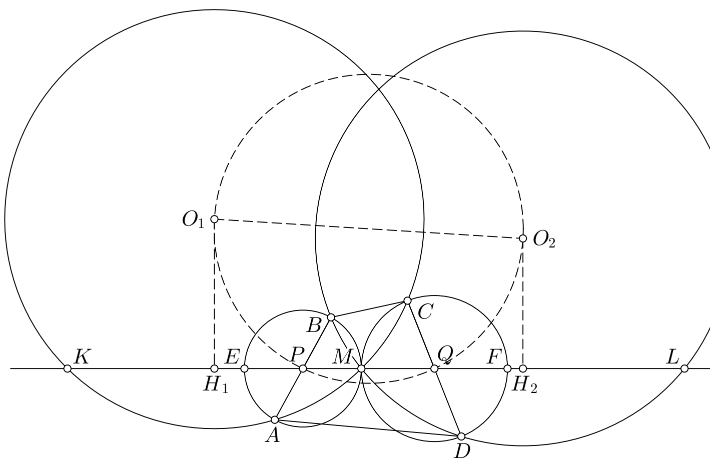
Step 3. Clearly, $KH_1 = H_1M$ and $LH_2 = H_2M$ (the radius perpendicular to a chord bisects it).

Step 4. We have $PH_1 = QH_2$, since the projection of the midpoint of O_1O_2 bisects both H_1H_2 and PQ (here we have again used the fact that the radius perpendicular to a chord bisects it). It finally follows that

$$\begin{aligned}|MK - ML| &= 2|MH_1 - MH_2| = 2|MP - MQ| \\ &= 2\left|\tfrac{1}{2}AB - \tfrac{1}{2}CD\right| = |AB - CD|.\end{aligned}$$

Second solution. Denote by ω_{AMB} the circle passing through A, M, and B, by ω_{AMC} the one through A, M, and C, and so on.

Let E be the intersection point of KL with ω_{AMB}, and let F be the intersection point of the same line and ω_{CMD} ($E, F \neq M$; see figure). Since EM is the diameter of the circle ω_{AMB}, we have $EM = AB$. It follows that $MK - AB = KE$. Similarly, $ML - CD = FL$. Therefore, it will suffice to show that $KE = FL$.



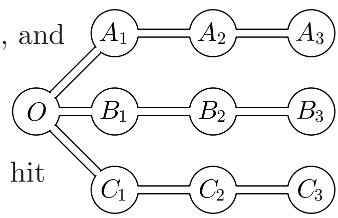
If the extension of AB meets ω_{AMC} at a point K', then $KE = K'B$. Indeed, since a circle is symmetric about any of its diameters, the reflection in the line PO_1 drawn through the centers of ω_{AMC} and ω_{AMB} takes both these circles into themselves. It follows that each of the common points A and M of these circles is taken under this reflection into the other. At the same time, points on the mirror line, in particular, P, stay in place. Therefore, the line AP is reflected into MP and the intersection K' of AP and the circle ω_{AMC} is reflected onto K, the intersection of MP and the same circle. Similarly, point E is reflected onto B. Hence segment KE is taken onto $K'B$, and so the two segments are congruent.

For the same reasons, if L' is the intersection point of the line CD and the circle ω_{BMD} distinct from D, then $FL = CL'$. Thus, it remains to prove that $BK' = CL'$.

Let L'' be the meet of CD and ω_{AMC} distinct from C. Considering the reflection in the line QO_1, joining the centers of circles ω_{AMC} and ω_{CMD} and taking each of them into itself, we see that $MK = CL''$. Similarly, $MK = AK'$. Therefore, $CL'' = AK'$. Suppose that the chords AK' and CL'' meet at N. Since they are congruent, the bisector of the angle ANL'' passes through the center of circle ω_{AMC}.

In the same way, it is proved that this bisector passes through the center of ω_{BMD} (if the extension of AK' meets ω_{BMD} at point K'', then it can be shown that $BK'' = DL'$). Therefore, the reflection in this bisector takes the circles ω_{AMC} and ω_{BMD} into themselves and the lines CL' and AK' into each other. Hence it takes the *segments* BK' and CL' intercepted by these circles on these lines into each other. Thus these segments are congruent, which completes the solution.

Problem 6. (a) Let's call the configuration in the statement of the problem a *tripus*: an octopus with three legs. We must prove that the tripus is safe. Label the shelters as in the figure.



We'll call the shelters O, A_2, B_2, and C_2 *even*, and the remaining ones *odd*. Let's assume the target starts in an even shelter. This assumption can only make things easier for the shooter, but we will prove that the target might avoid being hit even so.

From an even shelter the target can only run to an odd one, and vice versa. Therefore, after any odd number of shots the target is in an odd shelter and after an even number of shots, in an even shelter. This means that the player must fire his even shots at odd shelters and odd shots at even shelters.

We will prove that before each odd shot, the target can be in O and in one of two other even shelters (i. e., the shooter doesn't know in which of these three shelters the target is hiding).

The proof is by induction on the number of shots. Our statement is clearly true at the start. Suppose that before the $(2k-1)$-st shot the target can be in O and in two other even shelters — say A_2 and B_2.

<u>Case 1.</u> Suppose the $(2k-1)$-st shot is fired at O. Then before the next shot the target can be in any of A_1, A_3, B_1, and B_3. The $2k$-th shot must be fired at an odd shelter. It is readily seen that in all cases, before the $(2k+1)$-st shot the target can be in any of the shelters O, A_2, and B_2 again.

<u>Case 2.</u> Suppose that the $(2k-1)$-st shot is fired at A_2. Then before the following shot, the target can be in any of A_1, B_1, C_1, and B_3. If the $2k$-th shot is fired at A_1, then before the $(2k+1)$-st shot the target can be in O, B_2, and C_2; if the shot was fired at B_1, the possible locations of the target are O, A_2, and C_2; and if at C_1, then O, A_2, and B_2. If the shooter fires at one of the shelters A_3, B_3, C_3, the target can appear in any of four shelters. In any case, before the $(2k+1)$-st shot the target can be in O and two more even shelters. This completes the proof of our statement in case 2.

<u>Case 3.</u> Suppose the $(2k-1)$-st shot is fired at B_2. This case is completely similar to the previous one.

<u>Case 4.</u> Suppose the $(2k-1)$-st shot is fired at C_2. Then before the following shot, the target can appear in any of the shelters A_1, A_3, B_1, and B_3, and this case is similar to case 1.

Thus, before each odd shot, the target can be in several different shelters, and the shooter won't necessarily be able to hit it. It is clear from the previous analysis that the shooter can't hit it in an even shot either.

(b) If a configuration includes a cycle $A_1A_2 \dots A_n(A_1)$ of several shelters, it is safe. Indeed, the target *always* has two choices of where to go, so no strategy can possibly lead to a situation where the shooter *knows* where the target will be next.

Also, if some part of a configuration is safe by itself, the target can survive by staying only in this part; hence the entire system is safe as well. Therefore, if a configuration includes a cycle plus anything else, it does not satisfy the minimality condition in part (b) of the statement of the problem.

Let's show that any cycle is a minimal configuration. Imagine that one of the edges, say A_nA_1, is removed. What is left is a linear chain of shelters, $A_1A_2 \dots A_n$. The target is then doomed, if the shooter uses the following strategy: Fire successively at A_1, A_2, $\dots$, A_{n-1}. This necessarily hits the target if it started off in an odd-numbered shelter (for the proof, consider the moment when the distance from the target to the shelter hit at this moment is minimal). If still alive, at the end of these $n-1$ shots the target is either in an odd- or an even-numbered shelter, depending only on the parity of n. The shooter now goes back to A_1 (if n is even) or to A_2 (if n is odd), and again shoots the shelters in increasing order up to A_{n-1}. This second sweep is guaranteed to hit the target.

It remains to consider configurations without cycles. We'll show that the tripus is the only safe minimal configuration of this kind. We describe the strategy of the shooter, assuming there is neither a cycle nor a tripus. (We'll do this assuming all the shelters are connected together. If the configuration is disconnected, i. e., it consists of two or more components not joined by tunnels to one another, then the shooter must successively apply the strategy below to each component.)

Let a shelter with three or more tunnels radiating from it be called a *hub*. A tunnel starting from a certain shelter is said to be a *through-pass* if, having run through it, the target can run through two more tunnels without visiting the same shelter twice. For instance, the tunnel A_1A_2 in the tripus leading from the shelter A_1 is not a through-pass, whereas the tunnel A_2A_1 leading from A_2 is a through-pass. Finally, a shelter is called a *dead-end* if there is only one tunnel leading from it.

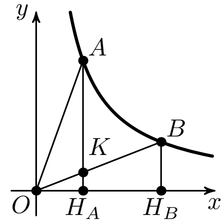
Since our configuration does not contain a tripus or a cycle, there are at most two through-pass tunnels leading from any shelter. Let us determine the first shelter to be attacked by the shooter. We take any hub. If there are two through-pass tunnels starting at this hub and leading to another hub each, we choose any of them and pass through it (and, perhaps, through a number of other tunnels) to the closest hub. If there is another through-pass tunnel leading from this new hub to another one, we pass through it to the next closest hub and proceed so until we arrive at a hub with a single through-pass tunnel issuing from it or two such tunnels one of which leads to a dead-end without passing through hubs on the way. In the first case, we move along any non-through-pass tunnel to a neighboring shelter; in the second case, we move through the through-pass tunnel leading to the dead-end and stop at the shelter adjacent with the dead-end. This determines the shelter at which the shooting must begin.

Let us divide all shelters into even and odd ones so that any tunnel joins shelters of different parity (this is possible, because there are no cycles). We'll explain how to fire in order to hit the target provided that initially he was hiding in a shelter of the same parity as the one at which the shooter fires first. (If, after this series of shots, the target survives, then his initial location was of different parity. Now the current parity of the target's location is known exactly.)

Suppose that the shooter fires at successive shelters one by one starting with the one chosen above and ending with a hub. Then the target is outside the bombarded linear part of the configuration. We have chosen the initial shelter so that at most one of the tunnels leading to other parts of the configuration is a through-pass. (Even if there are two through-pass tunnels, the second one leads to the part that has just been shelled.) Any non-through-pass tunnel leads either to a dead-end shelter or to a shelter from which one can get to a number of dead-ends or come back to the hub. In both cases, there is only one shelter beyond this tunnel whose parity is opposite to that of the hub in question. The same is the parity of the shelter to which the target moves after the shot at the hub. So if this shelter is beyond this tunnel, then the target will be hit by the shot at the shelter to which the tunnel leads. Otherwise, the shooter fires at the hub again, thus preventing the target from running over to the parts of the configuration that have already been "checked." Having checked all non-through-pass tunnels, the shooter sets on the only through-pass one, hits the shelter to which it leads, and then successively all shelters up to the nearest hub. Then the non-through-pass tunnels at this hub are checked in the same way as above, and so on. Thus the entire configuration can be checked.

We have shown that a configuration that contains no cycle and no tripus cannot be safe. Destruction of any tunnel in a tripus makes it unsafe, so a tripus is a minimal safe configuration. Finally, any configuration consisting of a tripus and something else is safe, but not minimal.

Level C



Problem 1. We can assume the x-coordinate of point A is less than that of B (see figure). Let K be the intersection of the segments AH_A and OB. The curvilinear triangle AKB is the intersection of the figures whose areas we must show to be equal. Therefore, the difference in area between the two figures is equal to the difference between the areas of the triangle OAK and the

quadrilateral H_AKBH_B. But this difference is zero:

$$S(OAK) - S(H_AKBH_B) = S(OAH_A) - S(OBH_B) = \tfrac{1}{2}\,OH_A \cdot AH_A - \tfrac{1}{2}\,OH_B \cdot BH_B = \tfrac{1}{2} - \tfrac{1}{2} = 0.$$

The last but one equation follows from fact that points A and B lie on the graph of the function $y = 1/x$.

Problem 2. Notice that

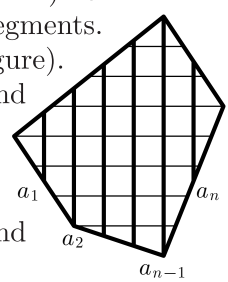
$$f(x) = (x+6)^2 - 6.$$

Then $f(f(x)) = \big(((x+6)^2-6)+6\big)^2-6 = (x+6)^4-6$, $f(f(f(x))) = (x+6)^8-6$ and so on. Finally,

$$f(f(f(f(f(x))))) = (x+6)^{32} - 6,$$

and the solution of the equation $(x+6)^{32} = 6$ is obvious: $x = -6 \pm \sqrt[32]{6}$.

Problem 3. We'll prove that each of the sums in the statement is equal to the area of the polygon (relative to the area of the grid square). Consider, for instance, the sum of lengths of the horizontal segments. Denote the lengths of these segments by $a_1, \dots, a_n$ (see figure). These segments divide the polygon into two triangles and $n-1$ trapezoids, the altitudes of all these subpolygons being equal to 1. By the familiar formula, the area of the i-th trapezoid is equal to $(a_i + a_{i+1})/2$ grid squares, and the areas of the triangles are equal to $a_1/2$ squares and $a_n/2$ squares. Hence the area of the polygon is equal to



$$\frac{a_1}{2} + \frac{a_1+a_2}{2} + \dots + \frac{a_{n-1}+a_n}{2} + \frac{a_n}{2} = a_1 + a_1 + \dots + a_n,$$

completing the proof.

Remarks. 1. The statement of the problem remains true for a nonconvex polygon as well. Furthermore, we can allow the polygon's sides to be horizontal or vertical if the lengths of horizontal and vertical segments on the boundary of the polygon are counted in the corresponding sums with a factor of $\frac{1}{2}$.

2. Here is an outline of an alternative solution. Each of the sums is *additive* in the following sense: if the polygon is cut into several pieces, then its "horizontal" or "vertical" sum is equal to the sum of the corresponding sums for all the pieces. But it is not difficult to see that the only additive function on the set of all polygons with integer vertices equal to 1 on a unit square and invariant under translations and central symmetries is the area.

3. Consider a sufficiently "good" set on the coordinate plane (for instance, a polygon or a convex set). Suppose that the length of its intersection with the line $x = a$ is equal to $f(a)$ and the length of its intersection with the line $y = b$ is equal to $g(b)$. Then the area of our set can be computed by the formulas

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} g(y)\,dy.$$

Our problem is a discrete analog of these formulas. Note that many statements in calculus have discrete analogs. See [651, Chapter 11, §4].

Problem 5. We explain how to perform the desired transformations in case (a) and (c). By deleting the first term, the sequence $\{a_n\}$ is transformed into the sequence $\{a_{n+1}\}$. Subtracting the first sequence from the second, we obtain the sequence $\{a_{n+1} - a_n\}$. We denote this transformation by T, and its m-fold iteration by T^m. Dividing the given sequence by itself, we obtain the useful sequence all of whose terms are ones; it will be denoted by I. Now let us show how to obtain the sequence $\{n\}$ in case (a):

$$\{n^2\} \xrightarrow{T} \{2n+1\} \xrightarrow{-I} \{2n\} \xrightarrow{/(I+I)} \{n\}.$$

Item (c) is harder. First, notice that if $P(n)$ is a polynomial in n of degree m, then the application of T to the sequence $\{P(n)\}$ yields the sequence $\{Q(n)\}$, where $Q(n)$ is a polynomial of degree $m-1$ (see Fact 22). Hence the application of T^{m-1} to $\{P(n)\}$ yields a polynomial of degree 1, that is, a sequence of the form $\{an+b\}$, and the application of T^{m+1} yields the zero sequence. We have

$$\left\{\frac{n^{2000}+1}{n}\right\} = \left\{n^{1999} + \frac{1}{n}\right\}.$$

The transformation T^{2000} applied to the term n^{1999} of this sequence yields 0. By induction, it can readily be seen that the application of T^{2000} to the second term yields

$$\frac{2000!}{n(n+1)\cdot\dots\cdot(n+2000)}.$$

Now it is not difficult to write out the desired chain of transformations:

$$\left\{\frac{n^{2000}+1}{n}\right\} \xrightarrow{T^{2000}} \left\{\frac{2000!}{n(n+1)\cdots(n+2000)}\right\} \xrightarrow{I/} \left\{\frac{n(n+1)\dots(n+2000)}{2000!}\right\}$$

$$\xrightarrow{\times(\overbrace{I+I+\dots+I}^{2000!\text{ times}})} \{n(n+1)\cdots(n+2000)\} \xrightarrow{T^{2000}} \{an+b\},$$

where a and b are integers and $a \neq 0$. The remaining operations are clear.

(b) We prove that in this case it is impossible to obtain the sequence $\{n\}$. First, all sequences that can be obtained from $\{n+\sqrt{2}\}$ are of the form

$$\left\{\frac{P(n+\sqrt{2})}{Q(n+\sqrt{2})}\right\},$$

where P and Q are polynomials with integer coefficients. (Indeed, the initial sequence is of this form. The term-by-term addition, subtraction, multiplication, or division of sequences of this form yields sequences of the same form again, and the deletion of a few initial terms is equivalent to the replacement of $P(x)/Q(x)$ by $P(x+r)/Q(x+r)$ for a certain positive integer r, which can be rewritten in this form by opening the brackets in the numerator and denominator.) Further, if the sequence $\{n\}$ can be represented in this form, the same is true for the sequence all of whose terms are equal to $\sqrt{2}$. But the relation $P(n+\sqrt{2})/Q(n+\sqrt{2}) = \sqrt{2}$ implies that the ratio of the leading coefficients of the polynomials P and Q is $\sqrt{2}$, which is impossible.

Problem 6. (a) Suppose that Greg names two sets of cards — the one he was given and a set of three other cards — and says, "I have one of these sets." Then Linda will get to know Greg's cards (because Linda's set is disjoint with Greg's set and necessarily intersects the second set called by Greg).

Now two situations are possible: if the second set named by Greg does not coincide with Linda's set, then Linda must name her set of cards and Greg's set of cards and say, "I have one of these sets."

If the second set named by Greg coincides with Linda's set, then this rule is no good, because Pat will compute the hidden card. So in this case Linda calls her set of cards plus any three other cards, so long as they don't form Greg's set.

After that each of the players will know the entire deal. On the other hand, Pat doesn't know anything definite. Indeed, three sets of cards were named: A, B, and C. Greg said, "I have either A or B"; Linda said, "I have either A or C" (the sets B and C have two cards in common, and other pairs of sets are disjoint). This means that either Greg has the set A and Linda the set C or Greg's set is B and Linda's set is A. Of course, these two deals are different, and even the hidden card cannot be determined.

(b) In this case method (a) does not work: knowing the hidden card, Pat will be able to exactly determine the deal. Let us number the cards from 0 to 6. Suppose that Greg and Linda in turn announce the remainders of the sums of numbers of their cards upon division by 7 (see Fact 6). Then they will get to know the deal: each of them must only add to his/her sum the other sum and find the remainder opposite to this total sum modulo 7 (i.e., the remainder that gives a sum divisible by 7 when added to Greg's and Linda's sums). This will be the number of the hidden card. After that Linda and Greg will readily figure out the deal.

Let us check that Pat learns nothing. Consider the card with number s. We show that it could have been dealt to Greg if he announced a sum a. It suffices to add to this card two other cards whose sum of numbers gives the same remainder as $a - s$ when divided by 7. It is readily seen (check this!) that there exist three different pairs of numbers with this property. The pair we need must not include the card with number s and Pat's card. Thus, two pairs at most can be left out of consideration; but in any case, at least one pair remains, and we complete Greg's set with this pair. Similar argument shows that Linda also can be dealt any given card.

Remark. Notice that the method of part (b) doesn't work in part (a): Nick will detect the hidden card.

Level D

Problem 1. Set $a = 2000m + n$, $b = 2000n + m$, and denote by d the greatest common divisor of a and b. Then d also divides the numbers

$$2000a - b = (2000^2 - 1)m \quad \text{and} \quad 2000b - a = (2000^2 - 1)n.$$

Since m and n are coprime, the number d is a divisor of $2000^2 - 1$ (see the remark). On the other hand, for $m = 2000^2 - 2000 - 1$, $n = 1$, we have

$$a = (2000^2 - 1)(2000 - 1), \quad b = 2000^2 - 1 = d.$$

Remark. We have used the following statement: if m and n are coprime, then $\gcd(dm, dn) = d$. This follows from the more general equality

$$\gcd(dx, dy) = d \gcd(x, y).$$

We outline the proof. Let $p_1, \ldots, p_n$ be all the prime factors of the numbers x, y, and d. Write the prime factorizations of these numbers:

$$x = p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n},$$
$$y = p_1^{l_1} p_2^{l_2} \ldots p_n^{l_n},$$
$$d = p_1^{m_1} p_2^{m_2} \ldots p_n^{m_n}.$$

(Some of the exponents k_i, l_i, and m_i can be equal to zero. Compare with the remark to Problem 95.B.4.)

Denote by $\min(p, q)$ the smaller of the numbers p and q. Then

$$\gcd(x, y) = p_1^{\min(k_1, l_1)} p_2^{\min(k_2, l_2)} \ldots p_n^{\min(k_n, l_n)}.$$

Similarly,

$$\gcd(dx, dy) = p_1^{\min(m_1+k_1, m_1+l_1)} p_2^{\min(m_2+k_2, m_2+l_2)} \ldots p_n^{\min(m_n+k_n, m_n+l_n)}.$$

It remains to notice that for any i, we have

$$\min(m_i + k_i, m_i + l_i) = m_i + \min(k_i, l_i).$$

Another proof can be obtained using the following fact: for any integer x and y, there exist integer a and b such that

$$ax + by = \gcd(x, y).$$

See [421], for instance.

Problem 2. *First solution.* Since the integral of the difference of functions is equal to the difference of integrals of these functions (see Fact 28), we have

$$\int_0^\pi \bigl(|\sin 1999x| - |\sin 2000x|\bigr)\,dx = \int_0^\pi |\sin 1999x|\,dx - \int_0^\pi |\sin 2000x|\,dx. \tag{1}$$

We'll show that

$$\int_0^\pi |\sin kx|\,dx = 2$$

for any positive integer k. This will imply that the right-hand side of (1) is zero.

The function $|\sin kx|$ is periodic with period π/k. This means that

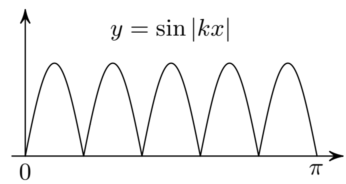
$$\int_0^{\pi} |\sin kx|\,dx = k\int_0^{\pi/k} |\sin kx|\,dx = k\int_0^{\pi/k} \sin kx\,dx. \qquad (2)$$

It is easy to compute the integral in the right-hand side by the change of variable $y = kx$ (see Fact 28):

$$\int_0^{\pi/k} \sin kx\,dx = \int_0^{\pi} \sin y\,\frac{dy}{k} = \frac{2}{k}.$$

Hence the integral (2) is equal to 2 for any k, and we are done.

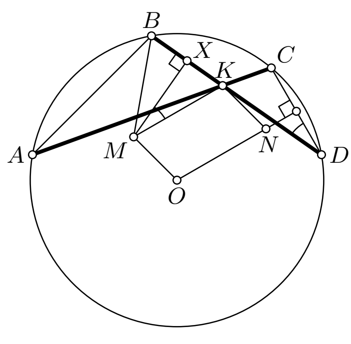
Second solution (*sketch*). The graph of the function $y = |\sin kx|$ on $[0,\pi]$ consists of k identical caps (see figure) obtained by compressing the graph of $y = \sin x$ on the same interval toward the y-axis by the factor k. In this process, the area under the graph also decreases by the factor k. Thus the area under the k caps remains the same for any k.

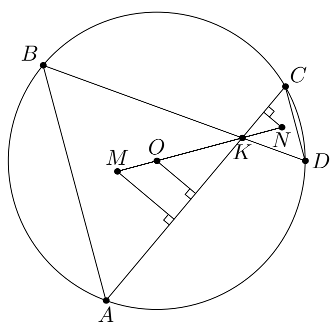


Problem 3. Denote by X the midpoint of KB (see left figure on the next page). Then

$$\angle KMX = \tfrac{1}{2}\angle KMB = \angle KAB = \angle KDC;$$

here the middle equation is based on the fact that the measure of an inscribed angle equals half the bend of its intercepted arc, and in the last equation we use the consequence of this theorem: inscribed angles subtended by the same arc are congruent. Clearly, MX is perpendicular to BD; therefore, KM is perpendicular to CD. We also have $ON \perp CD$; hence ON is parallel to KM. Similarly, $OM \parallel KN$.





If the points O, K, M, and N do not lie on the same line, then $OMKN$ is a parallelogram and $OM = KN$. Otherwise, we consider the orthogonal projections of segments OM and KN on AC (see figure above and to the right). Since the projections of points O, M, and N are the midpoints of segments AC, AK, and KC, respectively, the projections of OM and KN are equal to $\frac{1}{2}KC$. Since these segments lie on the same line, the equality of the lengths of projections implies the congruence of the segments themselves.

Problem 4. Consider the polynomial $P(x) = x^3 - x^2 - x - 1$. It has a root t greater than 1, since $P(1) < 0$ and $P(2) > 0$ (see the remark). Suppose that the lengths of the sticks are equal to t^3, t^2, and t. It follows that $t^3 = t^2 + t + 1 > t^2 + t$; by the triangle inequality, there is no triangle with such side lengths. After the longest stick is broken, we obtain three sticks of lengths t^2, t, and 1. Since the length ratios remain the same, the process will continue forever.

Remark. We've used the *intermediate value theorem*, which says that *if a function $P(x)$ is continuous on the interval $[a, b]$ and takes values of opposite sign at its endpoints — for instance, $P(a) < 0$ and $P(b) > 0$ — then there exists a point $c \in (a; b)$ such that $P(c) = 0$.*

We won't prove this theorem, or even define continuity; the reader can consult [651, Chapter 4]. We remark only that any polynomial is continuous.

Problem 5. (a) Let N be the number of players, and set $M = [N/2]$. The players that won the first M places will be called *strong* and the other players *weak*. Players with the same score are distributed between the two groups arbitrarily. Let X be the number of normal (not anomalous) games between strong and weak players. Strong players scored a total of $M(M-1)/2$ in games among themselves and no more than X in games against weak players. Denote by S_1 the total score of strong players and by S_2 the total score of weak players. Then

$$S_1 \leq \frac{M(M-1)}{2} + X, \qquad S_1 + S_2 = \frac{N(N-1)}{2}.$$

If all individual scores were equal, there would be no anomalous games. Therefore we can assume that there are two players with different scores. First, consider the case of an even N. In this case, there are equally many weak and strong players. Then, clearly, $S_1 > S_2$. It follows that

$$S_1 > \frac{S_1 + S_2}{2} = \frac{N(N-1)}{4},$$

whence

$$X \geq S_1 - \frac{M(M-1)}{2} > \frac{N(N-1)}{4} - \frac{M(M-1)}{2}.$$

Substituting $M = N/2$, we can readily ascertain that

$$X > \frac{N^2}{8} > \frac{N(N-1)}{8}.$$

Since the total number of games is $N(N-1)/2$, the portion of normal games is greater than $\frac{1}{4}$.

However, this proof doesn't work for odd N. In this case, we can proceed as follows: consider the *average* result S_1/M of a strong player. Clearly, it is greater than the average result of a weak player, and hence it is greater than the average over all players:

$$\frac{S_1}{M} > \frac{N(N-1)/2}{N}.$$

That is, $S_1 > M(N-1)/2$. Therefore,

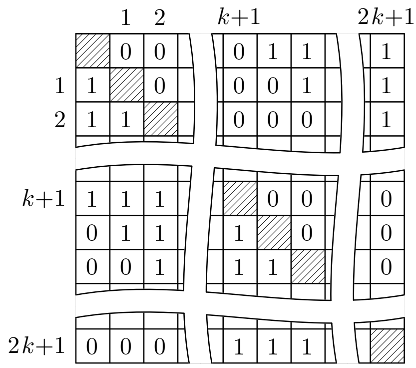
$$X \geq S_1 - \frac{M(M-1)}{2} > \frac{M(N-M)}{2} \geq \frac{N(N-1)}{8},$$

where the last inequality is checked by taking $M = (N-1)/8$.

Remark. The proof for odd N is good for even N as well.

(b) First, consider the tournament of $2k+1$ players in which each participant with a number $i \leq k$ lost to all participants with numbers $i+1, \dots, i+k$ and beat all the other participants, and each participant with a number $i > k$ beat the players with numbers $i-k, \dots, i-1$ and lost to the rest. Obviously, all players scored k points each.

Consider the tournament table (see figure; rows and columns correspond to players). It is not difficult to see that above the main diagonal, exactly



$k(k+1)/2$ positions out of $2k(2k+1)/2$ are occupied by 1s. Now imagine we clone each player, replacing him or her by a group of n players so that players from different groups play with each other with the same result as their originals, and players from the same group tie with each other. We obtain a new table in which all players have the same score again.

Let us correct this table so that the scores become different. We'll change results of players from the $(k+1)$-st group: in of their wins in the games with players from the $(k+1-i)$-th group must be replaced by ties so that the score of each player from the $(k+1)$-st group decreases by $i/2$, and that of each player from the $(k+1-i)$-th group increases by $i/2$. Similarly, in losses of players from the $(k+1)$-st group against $(k+1+i)$-th group are replaced by ties.

Then the first place in the tournament will be won by players from the first group, the second place will be taken by the second group, a nd so on. Let us count the number of anomalous games.

All games lost by players from groups with numbers $i \leq k$ are anomalous. All in all there are $kn^2 - in$ such games (recall that in games against players from the $(k+1)$-st group ended in a tie). For $i > k+1$, players from the i-th group lost $(2k+1-i)n^2$ anomalous games. Finally, players from the

$(k+1)$-st group lost $kn^2 - n\big(\frac{1}{2}k(k+1)\big)$ anomalous games; here the second term corresponds to anomalous losses replaced by ties.

Thus, the number of anomalous games is

$$\sum_{i=1}^{k}(kn^2 - in) + \sum_{i=k+2}^{2k+1}(2k+1-i)n^2 + kn^2 - n\frac{k(k+1)}{2}$$
$$= \frac{3k^2+k}{2}n^2 - k(k+1)n.$$

At the same time, the total number of games is $n(2k+1)(n(2k+1)-1)/2$. This means that, as k and n tend to infinity, the number of anomalous games grows as $\frac{3}{2}k^2n^2$, and the number of all games as $2k^2n^2$. The ratio of these numbers approaches $\frac{3}{4} > 0.7$ (see Fact 27). Therefore, for large enough n and k, this ratio will be greater than 70%. A reader who doesn't like limits can simply check that for $n = k = 20$ the fraction of anomalous games relative to all games is $\frac{235600}{335790} > 0.7$.

Problem 6. The desired configuration will be obtained from regular tetrahedra with the distance between opposite edges equal to the distance between the given planes. One edge of each tetrahedron will lie in one of the planes and the opposite one in the other plane. We'll think of the planes as horizontal.

Two tetrahedra can be positioned in such a way that an endpoint of the upper edge of one of them coincides with the midpoint of the upper edge of the other one, and the midpoint of the lower edge of the first tetrahedron coincides with an endpoint of the lower edge of the second, the edges of both tetrahedra in either plane being perpendicular. We can extend this construction to the entire layer to obtain a configuration in which each tetrahedron is surrounded by four others, as in the figure; two of them do not allow the tetrahedron to be pulled out of the layer upward, two other prevent it from moving downward.