Preface

This book presents materials that were used during the course of a year in one of the math circles for eighth graders organized by members of the mathematics faculty at Moscow State University. At different times and with different groups of students these materials have also been used in a math circle run at Moscow School Number 57, a mathematics "magnet" school. Some of the students had attended the circle the previous year, but since most were novices, the sessions were generally aimed at beginners. This does not mean that the sessions were easy, for even very simple problems can frustrate a beginner. Some minimal knowledge from school was expected, but much school material needed reviewing. For more advanced and experienced students, there were always extra problems available.

Students and their parents learn about math circles by word of mouth, from the Internet, or at math Olympiads. For instance, at the end of September in Moscow there is the Lomonosov Tournament, a multisubject competition for sixth through eleventh graders.[1] This is a remarkarble event. It is held on a Sunday in several Moscow colleges simultaneously, and it has recently spread to other cities. The rules and problems are the same for all sites. A student comes to one of the hosting sites and finds simultaneous contests in mathematics, math games, physics, biology, linguistics, history, astronomy, literature, and chemistry. Competing in any of these contests takes about an hour and can commence at virtually any moment during the Olympiad; the whole event takes about five hours. This gives students a chance, by moving from one room to another, to participate in several contests of their choice. Those who perform well in a given contest receive a winner certificate for this contest, and those who succeed in several contests get an all-around winner certificate.

But the certificates are not the most important goal; the real goal is to interest students in sciences and invite them to the circles. Students may dislike a given subject for all sorts of reasons — for example, if they happen to have a teacher who isn't very good — but when there are lots of contests happening simultaneously, their curiosity may force them to visit a contest, say, in math games, even if they are indifferent to mathematics. Their

[1] In the U.S., this would be seventh through twelfth graders.

attitude towards some subjects may change after the Lomonosov tournament since students see that every subject can be interesting. All the participants of the Tournament receive a brochure describing various Moscow circles.

Math circles have existed for more than a century at Moscow State University, and today they are combined under the name of "Little MechMat" — an allusion to the nickname for the university's renowned mechanics and mathematics school. There are two divisions: one with evening sessions and another by correspondence. The evening division of the Little MechMat comprises of circles for students in grades 6 through 11, although it isn't unusual for a fifth grader to attend a sixth graders circle. There is also a group for younger students. Anybody can participate in a math circle; there are no exams and all the circle sessions are free of charge.

The sessions are held in classrooms in the MechMat building on Saturday evenings when most of the rooms are vacant. In each classroom, there are 15 to 30 students and three to six instructors. Most instructors are university students and often they are former circle participants, but the head instructor is usually an experienced mathematician. The composition of the student group is fluid, as new students join all the time and some students drop out. Yet there is always a nucleus: a group of students who regularly attend the circle. The instructional staff does not change, except that some may not be able to attend every session. There is a winter break when students take exams and go on vacation.

The total number of participants in each grade from 6 through 8 is usually from 100 to 200 on any given evening. There are fewer students in the higher grades (9 through 11), since by this age many have become regular students of mathematics day schools (somewhat similar to "magnet schools" in the U.S.) and have demanding work loads there. Each grade has a head instructor who is responsible for designing the program for that particular grade. All instructors of a given grade may participate in developing class materials for that grade. Although individual sessions are held in several auditoriums, all sections of the same grade use the same handouts. One exception to this practice occurs when an instructor chooses to create his or her own individual course. In the higher grades such individual courses are more common, but the number of participants in each of them can be quite small.

The first class of the year is often given in written form because the organizers cannot anticipate how many students will show up and consequently don't know how many instructors will be necessary. Problems selected for the first session are not too difficult and do not require a lot of writing. Students are told that this is not a test: the main goal of the meeting is to find out what these students know, what they don't know, and what they need to learn. We tell them: the more you do not know, the better, because then there's more we can teach you. When they hear this, they break out smiling. We also tell the students that they can solve problems in any order, and that they should not be afraid of the prospect of solving just a few, and even if they can't solve any, it is still okay. If, on the other hand, some students solve all the problems quickly, they may be offered more challenging additional problems.

At the first meeting students also fill out a short questionnaire. At the next session each student finds his/her name, together with a specified classroom, on the list of circle participants. Students may change their classroom assignment if they wish.

In each subsequent session, a separate classroom is designated for that day's newcomers; later they are transferred to other classrooms. This way each regular classroom is periodically supplemented with newcomers. Session topics change quite often, so even students who join the circle in the middle of the year will see sets of problems with which they have a good chance of success.

Students vary considerably in strength and knowledge. There are several different approaches to dealing with this complication. Classrooms may be specifically designated for beginners and advanced students, with the level of the class materials accordingly different. This approach requires twice as much work, and quite often there are simply not enough resources to do this. Another possibility is to make only one version of the handout for all students, without sorting them into different levels. In this case every problem set must have very simple problems, more challenging ones, and also additional problems. This approach gives students an opportunity to move at their own pace, and it is quite manageable with several instructors in the room. Discussions at the board may also be done in a way that is useful for students at different levels.

Another wonderful approach is to put all new students into a big auditorium. There the students play fun games and solve easy problems that are carefully discussed. Those students who cope well with the easy problems are transferred to a regular math circle; the ones who struggle stay in the auditorium for newcomers. To be successful, this approach requires very good instructors for the newcomers.

The last sessions before the winter and summer breaks are very special. They may include a session on topological puzzles where students are tied with ropes and have to free themselves without cutting or untying the ropes. Another possibility is a competition which we call the Math Maze: it is described in detail starting on page 193, but here is the general idea.

Each student is given a map of the "maze" and the rules of the game. The map contains room numbers and their names, such as Mental Calculation, Games, Geometry, Logic, Combinatorics, or Puzzles. A student must visit all the rooms in any order and solve one problem in each room. The rooms are spread all over the building, so part of the fun is that some students like to race from one room to another. After a student has visited all the rooms on his map, he goes to the final auditorium where he can choose a book as a prize.

Winter break is quite long; sometimes we are able to shorten it by finding enthusiastic instructors willing to work with students even during their own final exams and vacations. But whatever the situation, before the break students are given a page with the Winter Competition problems, so that they do not forget the math circle. Students are expected to work on these problems over the break, write down their solutions and bring them to one of the first sessions after the break for grading. There are awards (usually math books). Everybody who turns in solutions gets one, but of course the higher the score, the greater the award.

The sessions at the Wednesday evening math circle at Moscow School Number 57, a day school with emphasis on math, are similar to the ones just described. About 100 students per grade participate; each room has 15 students and three to four instructors, including some older students from the school who are eager to help. The circle is for grades 7 and 8, though recently a new circle for sixth graders began. Their winter break is shorter since it coincides with the school break, but in the spring some of the circle sessions are replaced by interviews conducted in order to select students for special math classes at the eighth and ninth grade level. These interviews are not restricted to math circle participants: any interested student can come for an interview.

How do we run a circle session?

To begin with, all students get a handout with the main problems. Students read the problems and try to solve them. They can solve them in any order. An instructor might gently suggest that it could be a good idea to move down the list without spending too much time on any particular problem: if a problem takes too much time it might be better to leave it alone for a while and get back to it later.

A student who has a question or wants to discuss his or her solution simply raises a hand and an instructor comes by to talk. Students can explain their solutions orally, but it is useful to have at least some written notes with all necessary drawings and calculations. A conversation with an instructor is far from being a test. Sometimes a student has a solution but due to inexperience cannot articulate it, or he might totally misunderstand the problem, or he does not know some important facts, etc. The instructor's task is to help the student.

In the middle of the session there is a short break for anyone who needs it. Close to the break — either right before it or right after — solutions for the problems of the previous sessions are discussed with the whole class. The reason for doing it in the middle instead of the beginning of the session is to allow students to turn in those problems that they didn't finish in the previous meeting, but completed at home. While discussing old problems, some remarks concerning the current problem set might be appropriate; in some cases they might be offered even earlier if the topic seems to be hard for the students. It all depends on the head instructor, who might choose any moment to give a hint, or offer a funny problem to relax the class, or ask someone to come to the board and present her solution, or play a game.

Instructor explanations at the blackboard take only about 20 minutes out of a two-hour session. The rest of the time is devoted to students solving the problems and discussing them individually with instructors.

Students who are done with the main problems are provided with additional ones. Sometimes a student who, with plenty of time left, has solved all but one of the main problems and is stuck with the remaining one, might also be given additional problems, so as to be able to move on. At the end of the session the list of additional problems is given to anyone who asks.

There are always some students who solve very few or none of the problems, however easy the session might be. It is important that during any session an instructor comes to each student to give a hint or just to talk about the problems. A student who has not finished all the problems at the end of a session can solve them at home and discuss the solutions during the next session; however, there is no mandatory homework. Occasionally, hard additional problems are discussed several weeks after they were introduced, to give everyone a fair amount of time to work on them.

Each student's progress is registered in a special journal, where finished problems are marked with a plus sign. Children of this age love to compete: they are very happy to have many pluses, and they are disappointed to have just a few. We try not to focus children's attention on these signs because they are not the goal. It is the instructors who need these journals to measure the success of a session. It's better to register the results right during the class, since this gives a timely indication of which problems are harder, which require a hint, or pointing students into some particular direction.

Problem set structure: how problems are selected

The usual practice in regular schools is to create a lesson around a particular topic. For example, if today's topic is quadratic equations, then we explain some theory, give some examples, and solve some problems on quadratic equations. As a result, students quickly learn the new topic and can do standard exercises. This method definitely achieves results. However, I think that this method does not work well for math circles, especially for younger students.

One reason is that solving problems on the same topic for the whole session may be difficult and boring for a young student. Furthermore, if the instructor does not start a session with an explanation of how to solve problems on the new topic, a student who fails to figure out the key idea on his own can end up sitting through the whole session without solving a thing. On the other hand, if the instructor explains how to solve problems of a given type, it would leave less room for creativity, since it's enough just to remember what the teacher said and apply it to similar problems.

I choose a somewhat different approach. The progress is slower but, I believe, more reliable and interesting, and the method has some additional advantages. A typical handout looks like this:

The first problem is very easy. Sometimes it can be solved in different ways. One solution might not require any inventive thinking, just some straightforward work, while another might involve an interesting idea that would allow a student to solve the problem easily, quickly, and elegantly. For example, consider the following question: What is bigger, the sum of the first 50 odd natural numbers or the sum of the first 50 even natural numbers, and by how much? Some students will calculate both sums and compare them, but the answer can actually be made quite obvious without these cumbersome calculations. It is very useful to pose problems that have an intuitively obvious but wrong answer. For example: Is it possible for the product ab to be divisible by c^2 if neither a nor b is divisible by c? The "obvious" answer is, of course not. But in reality, ab can be divisible by c^2, or even by c^{100}. This problem makes an impression!

Actually, every problem set does have a particular topic, and approximately half of the problems are dedicated to it. Moreover, one medium-to-easy preliminary problem on this subject should be given in the previous handout, so that the students can think about this new type of problem beforehand. In the middle of the current session, when solutions of the previous handout are presented, this problem will be discussed. Thus, those students who have not yet figured out how to solve problems on the new topic will get a hint. The handout also contains some repetitive problems. I like asking the same questions dressed up differently. As an example, here are five questions that get at the same fundamental idea:

1. There are water taps in a school cafeteria. Each one can be open or closed. In how many ways can the water run in the cafeteria?

2. How many strings can one make of 0's and 1's, so that each string consists of 10 digits?

3. There are 10 apples growing on a tree. In how many ways can one pick some of them?

4. After school, 12 students decided to split into 2 groups, one to explore the city, and another to attend a class in computer science. In how many ways can they split?

5. The school cafeteria menu is always the same and consists of n different items. Peter wants to choose his breakfast differently every day; he can eat from 0 to n different items at a time. For how many days would he be able to do this?

Students often solve those problems as though they were completely new. But it makes me very happy if a student says "Wait! But we have already solved this problem!" This means that he has learned to see the essence of the problem and is not distracted by the appearance. Repeating problems that are essentially identical but presented in different formulations in a handout is also useful for those who missed some classes.

As already mentioned, each handout contains an easy, or medium-level problem, anticipating the topic of the next handout. One can give a problem on the same new topic in several subsequent sessions. Then the students themselves may gradually come up with a method to solve them. Later it will be easier and more natural for them to accept this method from the teacher when the topic is discussed in detail.

Sometimes students are given a whole sequence of problems, one per session, in which every problem helps to solve a later one. A good example is a sequence of problems about crossing the river or about catching a bus.

It is sometimes possible to give a hard, but captivating problem on a new topic to interest the students, but be sure to give them a lot of time for thinking. Then students will say at every new session, "Oh, let's discuss that problem — how can it be solved?" One can discuss the problem at the board together with the students, using brainstorming to obtain some intermediate result, and only discuss the complete solution several sessions later.

Additional problems based on either a new or an old topic are given to those who have mastered almost all of the main ones. They are designed for strong students and may be difficult or just interesting. They often require ingenuity and persistence to solve.

For younger students, it is good if many sessions have a mathematical game among the main problems. Kids can play it alone, with one another, or with an instructor. One can teach the children a lot of ideas with the help of the game problems. And what should one do if a student is persistently trying to submit a solution to a game problem based on a wrong strategy? Of course, one should play with him, but doing what? Beating him? That's not very good. One can give away the idea for the solution or disappoint the student. There is a wonderful technique that I learned from Nikolai Konstantinov. The instructor should adopt the strategy of the student and lose to him. If the student makes bad moves, the instructor may hint at the right ones and correct the wrong ones, but at the same time follow the strategy of the student at all times. As a result, the student wins, which is great, and at the same time is shown that his strategy is flawed.

Each session also has a geometric problem. They may be simple brainteasers, cutting problems, or problems of classical geometry requiring only minimal knowledge.

Children will not be bored with these sessions. They can use their previously acquired knowledge to solve problems on the earlier topics, or they may try to do problems on the new topic, or may try to solve problems that just require ingenuity.

An important advantage of this approach is that students will learn how to choose the method according to the problem. As an example of the need for this, I had a case where a student was taking an exam on three topics:

induction, combinatorics, and integers. The student received a problem and worked on it for a while unsuccessfully and then said, “I can’t solve this problem, because I can’t figure out what topic it’s about. Tell me what topic it’s in so I can solve it.” This was a bit depressing. I have to say that the student was in fact strong and is now successfully studying mathematics at a university.

It often does not occur to students that it is necessary to reason about the problem in order to solve it. In school, many students develop a habit of following a routine, often not understanding its meaning. One of my friends, university professor Gregory Rybnikov, told me that on a discrete mathematics exam he gave a reasonably good student the following nonstandard problem: “Prove that if $N > 1$, in any company of N people there will always be two who have the same number of friends in this company.” For a long time, the student could not figure out how to approach this problem, and then the teacher gave him a hint. “Let’s reason by contradiction: Let all the people in the company have a different number of friends. Note that each one in the company has no more than $N - 1$ friends.” “I see,” the student said, “the total number of people is N, and every one can have from 0 to $N - 1$ friends, so there is the same number N of possibilities. This means that someone has 0 friends, someone has one, someone has two, and so on until $N - 1$. Oh, this is an arithmetic progression! I can find its sum.” The sum of an arithmetic progression has absolutely nothing to do with this problem, but the student, who had already almost solved the problem, lost it. He got used to following routines — if there is an arithmetic progression, one should probably find its sum.

At the circle children are surprised to find out that problems can be solved through reasoning. Sometimes someone is so surprised that a simple argument without formulas can solve the problem that he asks, “Can we argue this way? Is this correct? Is it rigorous?” This happens, for example, with Problem 2.1 on page 5, about figures made of squares.

The goal of the math circle is not to explore the problems of a particular type or master a lot of factual material, but to interest students in mathematics, to show that mathematics is a beautiful and interesting science, to teach them to reason, and to distinguish a solution from a nonsolution.

Not all students can attend the circle regularly, and skipping a few sessions won’t impair their ability to work on a new handout. Although regular work is an important prerequisite in studying math, the most important one is probably interest. I know how to teach students who are interested — I can interest them even more — but there is almost nothing I can do with the indifferent ones. There are a lot of beautiful problems, and if the students can see that, they will succeed. To learn how to solve problems, one just needs to constantly work on them. Yuri Lysov, a student of Nikolai Konstantinov, believes that the meaning of a math circle is to show the students that they are able problem solvers.

Difficulties and Pitfalls

Various difficulties can arise. Inexperienced instructors sometimes cannot enforce discipline, and students talk too much and distract each other. Or a student may have been dragged into the circle by his parents, while he wants to run away and play soccer.

Sometimes the student-assistants cannot solve problems for sixth graders. It is very important that the problems be discussed with the instructors beforehand and that the instructors solve all the problems themselves, or at least learn the solutions for those they could not solve. The senior instructor should be able to solve all the problems and should have a great deal of knowledge and experience.

Students like asking questions, and this can lead to ones that are quite meaningful and difficult even if they are easy to formulate. The teacher also should not be afraid to say "I don't know" and find the answer by the next session. It can happen that the answer is too complex or even unknown. For example, it is unknown whether there is always a prime number in the interval from n^2 to $(n+1)^2$, and even the known fact that there is always a prime number in the interval from n to $2n$ is very difficult to prove.

Sometimes students visit the math circle after an Olympiad clamoring, "Tell us how to solve these problems." If that is the case, it is better to adjust the session's plan by including some of the problems from the Olympiad.

Students often infect each other with their enthusiasm, which is great, but sometimes there is not enough energy and the session is sluggish. It can happen that a problem set that appears accessible and interesting, at least for some students, is difficult or boring for others. When students find the problem set difficult, the head instructor must change the flow of the session and give out different problems. Another possibility is to start solving and discussing the problems together, dividing them into simpler parts, or giving hints. For example, if a game is played on a strip of size 1×100, the first piece of advice is to simplify the problem. What would be the answer if the strip has size 1×1, 1×2, or 1×3 instead? In finding answers to simpler problems, one can understand the general case. It is often very useful to look at special cases in order to try to guess the answer and then try to prove that the guess was correct. It is important that the students distinguish these intermediate guesses from a complete solution.

Sometimes the students, and even some teachers, believe that in each problem one must explain how they have found the solution. This is not always the case. For example, one might have a problem where one should come up with five numbers with a certain property. If the student just made a guess out of the blue but came up with five numbers with the desired property, this is wonderful: the ability to guess right is very useful in mathematics. Of course, to invent a general method for finding those numbers may also be important, and it is useful to talk about it, but that's no reason not to count this problem as solved. Sometimes the solution is just an inspired guess that comes from nowhere.

One should have a lot of patience in working with school children. A student might be initially inexperienced and understand very little, but it does not mean he is unable to learn mathematics. Once I had such a student, one who at first was only able to perform even the simplest tasks slowly and with great difficulty, but then decided to study seriously and worked all summer solving a whole collection of Olympiad problems. The student was admitted to a mathematics day school and became a successful mathematician.

One should not rush to cover a lot of topics and solve a lot of problems. If it is clear that the problems are too difficult, one must move more slowly, taking all the time that's needed.

Sometimes students who are not used to mathematics just do not understand the language we are using. Mathematics has a lot of conventions and tacit assumptions. Once I was trying to find of a problem in combinatorics for a specific student to solve, and came up with this: "How many ways are there to choose one person out of ten people?" The student's answer, 9, disconcerted me. "Why nine?" I asked. "I would not choose myself," replied the student. We were talking in different languages; he understood the simplest formulation in his own completely different way.

What a joy it is when students discover an insight for themselves. For example, one student tackled this problem: how many pairs can be made of two digits if the first digit in a pair can be any digit from 0 to 9, and the second digit in a pair can be any digit from 0 to 9? He started to write down the pairs. First he wrote down all pairs where both digits are the same, and so gradually got the answer 99 (he forgot the pair 00). That answer simply shocked him. For a while he was looking frantically at his notes. Then he exclaimed, "It's just all numbers from 1 to 99!" This student made a small mathematical discovery, and this is important, even if it took him a long time to come to it, and despite making a small mistake in the process.

Not everyone will become interested in mathematics, but if a student has the desire and perseverance, something good is bound to happen. It is also worth mentioning that this book contains problem sets of one particular math circle for one particular year. I changed almost nothing, although some sets turned out to be more complicated and some easier than planned. Lessons vary from year to year and from class to class. This compilation may be of great help to the instructor, but, of course, one should not just copy these lessons. Depending on the students, it may be necessary to change the topics, difficulty level, number of problems, etc. It is all in the hands of the teacher!

What's next?

At the end of the year, or at the end of a two- or three-year long course, students sometimes receive humorous "little math scholar" certificates. Attending the circle does not bring any formal benefits, but the acquired knowledge and skills help them a lot more than any certificates. There are some cases, however, when a document can come in handy. Once I was approached by a sixth grader who complained that he could not enroll in a particular very good library, because, they said, he was too young. The boy asked me for help. I wrote a letter, stating that the boy was a good student at "The Little MechMat" and asked for the student to be allowed to enroll in the library. An official paper with a Moscow State University seal did the job, and the boy was enrolled.

School children who finish one year of the circle may transfer to its continuation the next year. The instructors of The Little MechMat usually also transfer to the next grade and some of them organize groups for the continuing students in parallel with groups for newcomers and for those who did not have too much progress over the past year.

Many students get into mathematics day schools and start studying the subject more intensively. Of the students in these schools, sometimes half are former participants in math circles. The main education in math schools takes place at the formal lessons where attendance is mandatory. Yet it is important even in a mathematics magnet school to have lessons that students can attend at will. For all students of my math class who are interested, I lead a separate math circle, and a seminar based on the journal *Kvant*.

A mathematical circle in a mathematical school is usually attended by a few strong students since the regular school load is already difficult enough. The sessions are very informal. Often I just bring a collection of Olympiad problems and, right in the class, choose the problems that I like. The students can participate in the selection. The teacher can make a session on some topic, or introduce some interesting idea to the students. Sometimes, students themselves bring problems, and we solve them. Fans of geometry shout "Give us an interesting problem in geometry." Solutions are discussed with students individually at first, and then each one can present his solution at the board. Sometimes, if the problem is difficult, we solve it together. This is an exciting time for the children. After the circle we have tea with cookies.

Since I work in *Kvant*, I lead a seminar affiliated with this journal. Articles sent to the journal are subject to refereeing, but the referees are not only math experts — and one of the referees is a student. The student must understand the article and give a talk. At the time of the talk, which is attended by teachers and students, the speaker will answer questions from other students. We often think together on a difficult point. The seminar resembles a research seminar in that it is sometimes difficult to understand what the author had in mind. We sometimes find typos, errors, or obscure passages. We write a review for the author with recommendations on how to improve the paper. This is very important information, coming from prospective readers. The children see that they are doing important work; they determine whether an article will be published. Later, when they see the article in the journal, they feel quite encouraged. Sometimes there is even a dispute as to who will give a talk in the next seminar. Often there are too many students willing to give one.

Besides the obvious benefits for the journal, there is a tremendous advantage for the students as they learn how to read mathematical papers and understand them on their own. This is something that is lacking in a regular classroom. Students learn to speak and learn to listen. The seminar ends with the tea and further informal discussions. Not all students can give a good talk, but all try and gradually learn. In fact, the talks of some students are better than many of the lectures by university professors. A teacher may also base a seminar on old papers, which I often do. From 1990 through 2001, the journal *Quantum*, the American brother of *Kvant*, was published by the National Science Teachers Association and Springer-Verlag, and it can serve as a very good source for such work.

Where do the problems in this book come from?

Almost none of the problems in this book were invented by me; they were selected from a variety of sources, including various Olympiad and math circle collections, books, journals, and sometimes the oral folklore. Many problems entered the folklore so long ago that it does not seem possible to name the authors. My contribution is that I selected them, compiled them into the circle's sessions, and wrote solutions.

I hope that this book will be interesting to students who love math and are not afraid of difficulties, and maybe even help someone to fall in love with mathematics.