

Abstract

There are two main interrelated goals of this paper. Firstly we investigate the sums

$$S_N(\alpha, \gamma) := \sum_{n=1}^N \frac{1}{n \|n\alpha - \gamma\|}$$

and

$$R_N(\alpha, \gamma) := \sum_{n=1}^N \frac{1}{\|n\alpha - \gamma\|},$$

where α and γ are real parameters and $\|\cdot\|$ is the distance to the nearest integer. Our theorems improve upon previous results of W. M. Schmidt and others, and are (up to constants) best possible. Related to the above sums, we also obtain upper and lower bounds for the cardinality of

$$\{1 \leq n \leq N : \|n\alpha - \gamma\| < \varepsilon\},$$

valid for all sufficiently large N and all sufficiently small ε . This first strand of the work is motivated by applications to multiplicative Diophantine approximation, which are also considered. In particular, we obtain complete Khintchine type results for multiplicative simultaneous Diophantine approximation on fibers in \mathbb{R}^2 . The divergence result is the first of its kind and represents an attempt of developing the concept of ubiquity to the multiplicative setting.