Contents

Introduction to Subgroup Decomposition 1
The main theorem, slightly simplified 2
Rotationless versus $\mathrm{IA}_n(\mathbb{Z}/3)$ (Part II) 3
The main theorem, full version 4
The relative Kolchin theorem for $\mathsf{Out}(F_n)$ (Part II) 6
Geometric models (Part I) 8
Vertex group systems (Part I) 9
Weak attraction theory (Part III) 9
Relatively irreducible subgroups (Part IV) 11

Part I. Geometric Models 15

Introduction to Part I 17

Chapter 1. Preliminaries: Decomposing outer automorphisms 23
1.1. F_n and its subgroups, marked graphs, and lines 23
1.2. Subgroup systems carrying lines and other things 31
1.3. Attracting laminations 35
1.4. Principal automorphisms and rotationless outer automorphisms 36
1.5. Relative train track maps and CTs 39
1.6. Properties of Attracting Laminations 54

Chapter 2. Geometric EG strata and geometric laminations 65
2.1. Defining and characterizing geometric strata 65
2.2. Complementary subgraph and peripheral splitting 78
2.3. The laminations of a geometric stratum 81
2.4. Geometricity is an invariant of a dual lamination pair 91
2.5. Stabilizing its complement also stabilizes the surface 92
2.6. Preserving the free boundary circles 95

Chapter 3. Vertex groups and vertex group systems 103
3.1. Vertex group systems 103
3.2. Geometric models and vertex group systems 105

Part II. A relative Kolchin theorem 109

Introduction to Part II 111

Chapter 1. Statements of the main results 113

Chapter 2. Preliminaries 117

2.1. Polynomial growth relative to a free factor system. 117
2.2. F_n-trees 118
2.3. One-edge extensions: free factor systems versus graphs 119
2.4. Asymptotic data: attracting laminations, eigenrays, and twistors 120
2.5. Complete splittings rel G_r 125
2.6. Fixed subgroup systems 126

Chapter 3. An outline of the relative Kolchin theorem 129

Chapter 4. $\mathrm{IA}_n(\mathbb{Z}/3)$ periodic conjugacy classes 133
4.1. Reduction to: F_n is filled by the θ-periodic conjugacy classes 134
4.2. Each CT $f\colon G \to G$ representing ϕ is geometric/linear/fixed. 134
4.3. The case that $\cup\mathcal{L}(\theta)$ fills. 135
4.4. The case that $\cup\mathcal{L}(\theta)$ does not fill: a one-edge extension. 138
4.5. The case that $\cup\mathcal{L}(\theta)$ does not fill: conclusion. 140

Chapter 5. $\mathrm{IA}_n(\mathbb{Z}/3)$ periodic free factors 147
5.1. Reduction to one-edge extensions: Proposition 5.1 147
5.2. Relative Nielsen classes and the path set Γ 149
5.3. Proof that $\mathrm{IA}_n(\mathbb{Z}/3)$ periodic free factors are fixed: Theorem 3.1 157

Chapter 6. Limit Trees 161
6.1. Iteration of growers 162
6.2. Iteration of nongrowers 164

Chapter 7. Carrying asymptotic data: Proposition 3.4 169
7.1. Carrying eigenrays 169
7.2. The exponential growth digraph for twistors 170
7.3. Bouncing sequences 175

Chapter 8. Finding Nielsen pairs: Proposition 3.7 179

Part III. Weak Attraction Theory 189

Introduction to Part III 191

Chapter 1. The nonattracting subgroup system 195
1.1. The nonattracting subgroup system $\mathcal{A}_{\mathrm{na}}(\Lambda^+_\phi)$ 196
1.2. Applications and properties of the nonattracting subgroup system. 201
1.3. Weak convergence and malnormal subgroup systems. 203

Chapter 2. Nonattracted lines 205
2.1. Characterizing nonattracted lines: Theorem G 205
2.2. Concatenating special nonattracted lines 208
2.3. Proof of Theorem H 212
2.4. Nonattracted lines of EG height: the nongeometric case. 213
2.5. Nonattracted lines of EG height: the geometric case. 216
2.6. General nonattracted lines: Proof of Theorem G 229

Part IV. Relatively irreducible subgroups 235

Introduction to Part IV 237

Chapter 1. Ping-pong on geodesic lines 241
1.1. Finding attracting laminations by the "three over one" criterion 241
1.2. The ping-pong argument 242

Chapter 2. Proof of Theorem C 249
2.1. Reduction to Theorem I 249
2.2. Constructing a conjugator 250
2.3. Driving down $\mathcal{A}_{\mathrm{na}}(\Lambda^{\pm}_{\phi})$ 252
2.4. Driving up $\mathcal{F}_{\mathrm{supp}}(\mathcal{F}, \Lambda^{\pm}_{\phi})$: Proof of Theorem I 256
2.5. Relatively geometric irreducible subgroups: Theorem J 261

Chapter 3. A filling lemma 267

Bibliography 271

Index 275