Abstract

This article investigates structural, geometrical, and topological characterizations and properties of weakly modular graphs and of cell complexes derived from them. The unifying themes of our investigation are various "nonpositive curvature" and "local-to-global" properties and characterizations of weakly modular graphs and their subclasses. Weakly modular graphs have been introduced as a far-reaching common generalization of median graphs (and more generally, of modular and orientable modular graphs), Helly graphs, bridged graphs, and dual polar graphs occurring under different disguises (1–skeletons, collinearity graphs, covering graphs, domains, etc.) in several seemingly-unrelated fields of mathematics:

- Metric graph theory
- Geometric group theory
- Incidence geometries and buildings
- Theoretical computer science and combinatorial optimization

We give a local-to-global characterization of weakly modular graphs and their subclasses in terms of simple connectedness of associated triangle-square complexes and specific local combinatorial conditions. In particular, we revisit characterizations of dual polar graphs by Cameron and by Brouwer-Cohen. We also show that (disk-)Helly graphs are precisely the clique-Helly graphs with simply connected clique complexes. With l_1–embeddable weakly modular and sweakly modular graphs we associate high-dimensional cell complexes, having several strong topological and geometrical properties (contractibility and the CAT(0) property). Their cells have a specific structure: they are basis polyhedra of even $\triangle$–matroids in the first case and orthoscheme complexes of gated dual polar subgraphs in the second case. We resolve some open problems concerning subclasses of weakly modular graphs: we prove a Brady-McCammond conjecture about CAT(0) metric on the orthoscheme