Abstract

We obtain local boundedness and maximum principles for weak subsolutions to certain infinitely degenerate elliptic divergence form inhomogeneous equations, and also continuity of weak solutions to homogeneous equations. For example, we consider the family $\{f_\sigma\}_{\sigma>0}$ with

$$f_\sigma(x) = e^{-\left(\frac{1}{|x|}\right)^\sigma}, \qquad -\infty < x < \infty,$$

of infinitely degenerate functions at the origin, and show that all weak solutions to the associated infinitely degenerate quasilinear equations of the form

$$\mathrm{div} A(x,u)\,\mathrm{grad} u = \phi(x), \quad A(x,z) \sim \begin{bmatrix} I_{n-1} & 0 \\ 0 & f(x_1)^2 \end{bmatrix},$$

with rough data A and ϕ, are locally bounded for admissible ϕ provided $0 < \sigma < 1$. We also show that these conditions are *necessary* for local boundedness in dimension $n \geq 3$, thus paralleling the known theory for the smooth Kusuoka-Strook operators $\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + f_\sigma(x)^2 \frac{\partial^2}{\partial x_3^2}$. We also show that subsolutions satisfy a maximum principle for admissible ϕ under a very mild restriction on the degeneracy. Finally, continuity of solutions is derived in the homogeneous case $\phi \equiv 0$ under a more stringent assumption on the degeneracy, namely that $f \geq f_{3,\sigma}$ for $0 < \sigma < 1$ where

$$f_{3,\sigma}(x) = |x|^{\left(\ln\ln\ln \frac{1}{|x|}\right)^\sigma}, \qquad -\infty < x < \infty.$$

As an application we obtain weak hypoellipticity (i.e. smoothness of all weak solutions) of certain *infinitely* degenerate quasilinear equations in the plane

$$\frac{\partial u}{\partial x^2} + f(x, u(x,y))^2 \frac{\partial u}{\partial y^2} = 0,$$

with smooth data $f(x,z) \gtrsim f_{3,\sigma}(x)$ where $f(x,z)$ has a sufficiently mild nonlinearity and degeneracy.

In order to prove these theorems, we first establish abstract results in which certain Poincaré and Orlicz Sobolev inequalities are assumed to hold. We then develop subrepresentation inequalities for control geometries in order to obtain the needed Poincaré and Orlicz Sobolev inequalities.