Abstract

A number of significant properties of C*-algebras can be expressed in continuous logic, or at least in terms of definable (in a model-theoretic sense) sets. Certain sets, such as the set of projections or the unitary group, are uniformly definable across all C*-algebras. On the other hand, the definability of some other sets, such as the connected component of the identity in the unitary group of a unital C*-algebra, or the set of elements that are Cuntz–Pedersen equivalent to 0, depends on structural properties of the C*-algebra in question. Regularity properties required in the Elliott programme for classification of nuclear C*-algebras imply the definability of some of these sets. In fact any known pair of separable, nuclear, unital and simple C*-algebras with the same Elliott invariant can be distinguished by their first-order theory. Although parts of the Elliott invariant of a classifiable (in the technical C*-algebraic sense) C*-algebra can be reconstructed from its model-theoretic imaginaries, the information provided by the theory is largely complementary to the information provided by the Elliott invariant. We prove that all standard invariants employed to verify non-isomorphism of pairs of C*-algebras indistinguishable by their K-theoretic invariants (the divisibility properties of the Cuntz semigroup, the radius of comparison, and the existence of finite or infinite projections) are invariants of the theory of a C*-algebra.

Many of our results are stated and proved for arbitrary metric structures. We present a self-contained treatment of the imaginaries (most importantly, definable sets and quotients) and a self-contained description of the Henkin construction of generic C*-algebras and other metric structures.

Our results readily provide model-theoretic reformulations of a number of outstanding questions at the heart of the structure and classification theory of nuclear C*-algebras. The existence of counterexamples to the Toms–Winter conjecture and some other outstanding conjectures can be reformulated in terms of the existence of a model of a certain theory that omits a certain sequence of types. Thus the existence of a counterexample is equivalent to the assertion that the set of counterexamples is generic. Finding interesting examples of C*-algebras is in some cases reduced to finding interesting examples of theories of C*-algebras. This follows from one of our main technical results, a proof that some non-elementary (in model-theoretic sense) classes of C*-algebras are 'definable by uniform families of formulas.' This was known for UHF and AF algebras, and we extend the result to C*-algebras that are nuclear, have nuclear dimension $\leq n$, decomposition rank $\leq n$, are simple, are quasidiagonal, Popa algebras, and are tracially AF. We show that some theories of C*-algebras do not have nuclear models. As an application of the model-theoretic vantage point, we give a proof that various properties of C*-algebras are preserved by small perturbations in the Kadison–Kastler distance.