

# Contents

Chapter 1. Introduction	1
Chapter 2. Continuous model theory	7
2.1. Preliminaries	7
2.2. Theories	10
2.3. Ultraproducts	13
2.3.1. Atomic and Elementary Diagrams	14
2.4. Elementary classes and preservation theorems	15
2.5. Elementary classes of $C^*$ -algebras	20
2.5.1. Abelian algebras	21
2.5.2. Non-abelian algebras	21
2.5.3. Real rank zero again	21
2.5.4. $n$ -subhomogeneous	21
2.5.5. Non- $n$ -subhomogeneous algebras	22
2.5.6. Tracial $C^*$ -algebras	22
2.5.7. $C^*$ -algebras with a character	22
2.6. Downward Löwenheim-Skolem	22
2.7. Tensorial absorption and elementary submodels	24
2.7.1. Strongly self-absorbing $C^*$ -algebras	24
2.7.2. Stable algebras	25
Chapter 3. Definability and $A^{eq}$	27
3.1. Expanding the definition of formula: definable predicates and functions	27
3.1.1. Definable predicates	27
3.2. Expanding the definition of formula: definable sets	27
3.3. Expanding the language: imaginaries	32
Countable products	32
Definable sets	33
Quotients	33
$M^{eq}$ and $T^{eq}$	34
3.4. The use of continuous functional calculus	35
3.5. Definability of traces	36
3.5.1. Definability of Cuntz–Pedersen equivalence	38
3.6. Axiomatizability via definable sets	40
3.6.1. Projectionless and unital projectionless	40
3.6.2. Real rank zero revisited	41
3.6.3. Infinite $C^*$ -algebras	41
3.6.4. Finite and stably finite algebras	41
3.7. Invertible and non-invertible elements	41

3.8.	Stable rank	43
3.9.	Real rank	45
3.10.	Tensor products	47
3.11.	$K_0(A)$ and $A^{\text{eq}}$	48
3.12.	$K_1(A)$ and $A^{\text{eq}}$	50
3.13.	Co-elementarity	52
3.13.1.	Abelian algebras	53
3.13.2.	Infinite algebras	53
3.13.3.	Algebras containing a unital copy of $M_n(\mathbb{C})$	53
3.13.4.	Definability of sets of projections	53
3.13.5.	Stable rank one	54
3.13.6.	Real rank zero	54
3.13.7.	Purely infinite simple $C^*$ -algebras	54
3.14.	Some non-elementary classes of $C^*$ -algebras	54
Chapter 4. Types		57
4.1.	Types: the definition	57
4.1.1.	Types as sets of conditions	57
4.2.	Beth definability	58
4.3.	Saturated models	60
4.3.1		60
4.3.2		62
4.4.	MF algebras	62
4.5.	Approximately divisible algebras	63
Chapter 5. Approximation properties		65
5.1.	Nuclearity	65
5.2.	Completely positive contractive order zero maps	65
5.3.	Nuclear dimension	66
5.4.	Decomposition rank	67
5.5.	Quasidiagonal algebras	67
5.6.	Approximation properties and definability	67
5.7.	Approximation properties and uniform families of formulas	68
5.7.1.	Uniform families of formulas	68
5.8.	Nuclearity, nuclear dimension and decomposition rank: First proof	70
5.9.	Nuclearity, nuclear dimension and decomposition rank: Second proof	73
5.10.	Simple $C^*$ -algebras	76
5.11.	Popa algebras	78
5.12.	Simple tracially AF algebras	78
5.13.	Quasidiagonality	80
5.14.	An application: Preservation by quotients	80
5.15.	An application: Perturbations	81
5.16.	An application: Preservation by inductive limits	84
5.17.	An application: Borel sets of $C^*$ -algebras	84
Chapter 6. Generic $C^*$ -algebras		87
6.1.	Henkin forcing	88
6.2.	Infinite forcing	90
6.3.	Finite forcing	93

6.4.	$\forall\exists$ -axiomatizability and existentially closed structures	95
6.5.	Strongly self-absorbing algebras	97
6.6.	Stably finite, quasidiagonal, and MF algebras	98
Chapter 7.	$C^*$ -algebras not elementarily equivalent to nuclear $C^*$ -algebras	101
7.1.	Exact algebras	101
7.2.	Definability of traces: the uniform strong Dixmier property	102
7.3.	Elementary submodels of von Neumann algebras	105
Chapter 8.	The Cuntz semigroup	107
8.1.	Cuntz subequivalence	107
8.2.	Strict comparison of positive elements	110
8.3.	The Toms–Winter conjecture	111
8.4.	Radius of comparison	112
Appendix A.	$C^*$ -algebras	113
Bibliography		117
Index		125