Abstract

In recent years it has been noted that a number of combinatorial structures such as real and complex hyperplane arrangements, interval greedoids, matroids and oriented matroids have the structure of a finite monoid called a left regular band. Random walks on the monoid model a number of interesting Markov chains such as the Tsetlin library and riffle shuffle. The representation theory of left regular bands then comes into play and has had a major influence on both the combinatorics and the probability theory associated to such structures. In a recent paper, the authors established a close connection between algebraic and combinatorial invariants of a left regular band by showing that certain homological invariants of the algebra of a left regular band coincide with the cohomology of order complexes of posets naturally associated to the left regular band.

The purpose of the present monograph is to further develop and deepen the connection between left regular bands and poset topology. This allows us to compute finite projective resolutions of all simple modules of unital left regular band algebras over fields and much more. In the process, we are led to define the class of CW left regular bands as the class of left regular bands whose associated posets are the face posets of regular CW complexes. Most of the examples that have arisen in the literature belong to this class. A new and important class of examples is a left regular band structure on the face poset of a CAT(0) cube complex. Also, the recently introduced notion of a COM (complex of oriented matroids or conditional oriented matroid) fits nicely into our setting and includes CAT(0) cube complexes and certain more general CAT(0) zonotopal complexes. A fairly complete picture of the representation theory for CW left regular bands is obtained.