Abstract

The L^p-Brunn–Minkowski theory for $p \geq 1$, proposed by Firey and developed by Lutwak in the 90's, replaces the Minkowski addition of convex sets by its L^p counterpart, in which the support functions are added in L^p-norm. Recently, Böröczky, Lutwak, Yang and Zhang have proposed to extend this theory further to encompass the range $p \in [0, 1)$. In particular, they conjectured an L^p-Brunn–Minkowski inequality for origin-symmetric convex bodies in that range, which constitutes a strengthening of the classical Brunn-Minkowski inequality. Our main result confirms this conjecture locally for all (smooth) origin-symmetric convex bodies in $\mathbb{R}^n$ and $p \in [1 - \frac{c}{n^{3/2}}, 1)$. In addition, we confirm the local log-Brunn–Minkowski conjecture (the case $p = 0$) for small-enough C^2-perturbations of the unit-ball of ℓ_q^n for $q \geq 2$, when the dimension n is sufficiently large, as well as for the cube, which we show is the conjectural extremal case. For unit-balls of ℓ_q^n with $q \in [1, 2)$, we confirm an analogous result for $p = c \in (0, 1)$, a universal constant. It turns out that the local version of these conjectures is equivalent to a minimization problem for a spectral-gap parameter associated with a certain differential operator, introduced by Hilbert (under different normalization) in his proof of the Brunn–Minkowski inequality. As applications, we obtain local uniqueness results in the even L^p-Minkowski problem, as well as improved stability estimates in the Brunn–Minkowski and anisotropic isoperimetric inequalities.