Abstract

In this monograph, we review the theory and establish new and general results regarding spreading properties for heterogeneous reaction-diffusion equations:

$$\partial_t u - \sum_{i,j=1}^{N} a_{i,j}(t,x)\partial_{ij} u - \sum_{i=1}^{N} q_i(t,x)\partial_i u = f(t,x,u).$$

These are concerned with the dynamics of the solution starting from initial data with compact support. The nonlinearity f is of Fisher-KPP type, and admits 0 as an unstable steady state and 1 as a globally attractive one (or, more generally, admits entire solutions $p^{\pm}(t,x)$, where p^- is unstable and p^+ is globally attractive). Here, the coefficients $a_{i,j}, q_i, f$ are only assumed to be uniformly elliptic, continuous and bounded in (t,x). To describe the spreading dynamics, we construct two non-empty star-shaped compact sets $\underline{\mathcal{S}} \subset \overline{\mathcal{S}} \subset \mathbb{R}^N$ such that for all compact set $K \subset \text{int}(\underline{\mathcal{S}})$ (resp. all closed set $F \subset \mathbb{R}^N \backslash \overline{\mathcal{S}}$), one has $\lim_{t\to+\infty} \sup_{x\in tK} |u(t,x) - 1| = 0$ (resp. $\lim_{t\to+\infty} \sup_{x\in tF} |u(t,x)| = 0$).

The characterizations of these sets involve two new notions of generalized principal eigenvalues for linear parabolic operators in unbounded domains. In particular, it allows us to show that $\overline{\mathcal{S}} = \underline{\mathcal{S}}$ and to establish an *exact* asymptotic speed of propagation in various frameworks. These include: almost periodic, asymptotically almost periodic, uniquely ergodic, slowly varying, radially periodic and random stationary ergodic equations. In dimension N, if the coefficients converge in radial segments, again we show that $\overline{\mathcal{S}} = \underline{\mathcal{S}}$ and this set is characterized using some geometric optics minimization problem. Lastly, we construct an explicit example of non-convex expansion sets.